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METATROPIC MATERIALS: A THEORY FOR PERMANENT MEMORY EFFECTS

by

J. E. Fitzgerald and R. Rubenstein

FINAL REPORT
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Air Force Office of Scientific Research
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Salt Lake City, Utah 84112

August 1976

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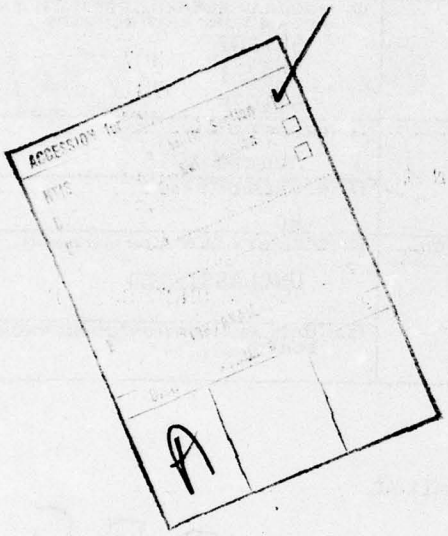

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[2]AFOSR Post Doctoral Fellow, University of Utah and Instructor, Applied Math, Georgia Institute of Technology.

ABSTRACT

A preliminary, mathematical paper [4] set forth the theoretical basis of permanent memory behavior in a restricted class of polymeric solids. The present report expands considerably upon [4] and sets forth a 3-dimensional theory of permanent memory effects, Metatropic Materials, applicable to heterogeneous solids.

The change in the value of the various moduli as well as the changes in the isotropy group, i.e. crystal class, of the stress response caused by a past sequence of strain histories is given a general structure.

The nature and sign of the residual stresses, permanent sets, and the isotropy group are deduced from an operational definition of metatropic effects. These include damage effects, strain hardening, and aging.

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NOMENCLATURE, CHAPTER I

C, $C^{(1)}$, $C^{(2)}$	Modulus tensors
D	Damage tensor
e	Infinitesimal strain tensor
G	Configuration
G_i	Kernel
P	Strain history
S	Stress
$\mathcal{m}$	Metatropic function
σ, τ	States

CHAPTER I: OVERVIEW

1.1 Introduction

Various theories of damage, permanent set, strain hardening and residual stress have appeared in the literature over the past 100 years. Most notable of the earlier works were those of Lord Kelvin [1] and E. Warburg [2]. Kelvin made the observation that the present state of a material element depends at any instant on its previous states as well as on the stimuli which obtain at the instant. He was, in this context, referring to permanent, irreversible changes in the material and not just to the viscoelastic or fading memory aspects so much in vogue today.

Warburg, in a series of experiments, demonstrated two very important phenomena particularly applicable to observations on filled and fibre reinforced polymers;

(1) A material after moderate shear straining exhibits a decrease in its tensile modulus and an increase in the value of its compressive modulus and

(2) An initially isotropic material after shear straining becomes essentially orthorhombic in its subsequent crystal class mechanical response.

Warburg was examining three dimensional permanent memory effects, and observing changes in both the values of the various moduli as well as changes in the isotropy group of the material elements.

An excellent review by Bartenev and Zuyez [3] points out rather strongly that "---creep and incipient failure proceed simultaneously---. The elementary act of failure precedes the elementary act of creep". They are referring to what we herein term Permanent Memory effects. Bartenev also stresses the great lack of true three-dimensional theory applicable to polymers.

It is the purpose of this paper to determine from a phenomenological viewpoint the assumptions necessary to formulate a three-dimensional framework for Permanent Memory effects and to examine and set forth the necessary mechanical consequences of such a generalized theory with respect to moduli, permanent set, residual stresses, and changes in the isotropy group of the material element.

1.2 Metatropic Materials

The senior author [4] set forth an initial attempt with the above mentioned purpose in 1974. At that time, the effect of prior strain histories on the istoropy group, i.e., crystal class, of the filled polymers tested was not included nor were the conclusions relative to permanent set and residual stresses fully formulated. Damage was the term then used to label these permanent memory phenomena and included the rather unsatisfactory concept of positive and negative damage effects. Other authors [5] attempting to correct the nomenclature problem coined the equally unsatisfactory term "divagation" meaning material whose properties wander about.

Herein we propose the term Metatropic Materials defined as "materials which change or tend to change their properties in a specified manner (meta-) and which produce or tend to produce a more highly organized state (-tropic) in response to the sequence of past stimuli".

While metamorphic might seem a similarly suitable term, its historic use in geology could confuse the issue.

The absence of the metatropic effects results in Equitropic materials whose isotropy group is invariant after any imposed strain history. This class of materials includes the usual linear or nonlinear viscoelastic solids and fluids as well as Noll's [6] semi-elastic materials.

1.3 The Metatropic Function

We denote the metatropic function, $\mathcal{M}$ and normalize its value to unity in the virgin state of the material.

It can be readily shown that the metatropic function, whether signifying a damaged material, $\mathcal{M} < 1$ or a strain-hardened material, $\mathcal{M} > 1$, can only be determined in general if

(1) The two samples being compared are in the same configuration and

(2) Both are completely relaxed with respect to creep or stress relaxation effects.

As an example, consider two samples of an ideal linearly viscoelastic material wherein $\mathcal{M} = 1$ for all states. Let σ, τ denote the

state of each sample respectively and let them each be in identical configurations, G.

Noll [6] defines equality of state operationally, that is, given two material elements in the same configuration, G, subject to a deformation (strain) process, P, if the observed tensorial stress outputs, S, are such that

$$S(\sigma,P) = S(\tau,P)$$

for every process, P, then we say that the states are equal, i.e., $\sigma = \tau$. If inequality holds for any P, then the states σ,τ are unequal.

If the material element in the state σ is one wherein the effects of a previous strain history have not fully relaxed and, say, τ represents the virgin, relaxed state then obviously the stress responses will not be equal even for our linearly viscoelastic model.

Thus, in order to determine whether the states σ and τ differ with a permanent metatropic change, all comparisons must be carried out from fully relaxed states in identical configurations.

We wish to associate the primitive concept of damage with some sort of weakening effect and strain-hardening with some sort of strengthening effect. The inequality of Fitzgerald [4] embodies these physically desirable characteristics as follows:

Definition 1

For states σ,τ of two fully relaxed material elements in identical

configurations, G, we say that the metatropic function $\mathcal{m}(\sigma) \leq \mathcal{m}(\tau)$ if and only if the stress response functional $S(\sigma,P) \leq S(\tau,P)$ for every deformation process which starts from the configuration, G.

Mathematically, we have defined a partial ordering on $\mathcal{m}$ through an isomorphism. States which obey D1 are called comparable. States which do not obey D1 are called incomparable and denoted by $\sigma || \tau$.

For uniaxial tests, all states are comparable. For general three-dimensional strain histories, however, the tensorial nature of stress can produce incomparable states.

1.4 Tensorial Character of $\mathcal{m}$

Let us examine the tensorial character of $\mathcal{m}$ from an heuristic point of view. For fully relaxed states in identical configurations subject to slow, infinitesimal strains, e_{kl} rather than general deformation processes, P, the linear elastic relation is

$$S_{ij} = C_{ijkl}\, e_{kl}$$

wherein C_{ijkl} is a fourth order tensor in general.

The metatropic function, $\mathcal{m}$, induces a change in the material coefficient values of the elasticities, C as well as possibly changing the isotropy group or crystal class of C. Thus the linear transformation from an initial elasticity $C^{(1)}$ to a changed elasticity $C^{(2)}$ is expressed by

$$C^{(2)}_{ijkl} = \mathcal{M}_{ijklmnop} C^{(1)}_{mnop}$$

with $\mathcal{M}$ an eighth order tensor.

Taking into account the usual symmetries of C, this linear approximation to $\mathcal{M}$ yields an expression requiring, in general, the evaluation of 231 constants. The impracticability of experimentally determining these constants is obvious.

Since experimental results on filled amorphous polymers show a certain "smearing" with respect to crystal class axes after a damaging history, it becomes practicable to restrict the form of $\mathcal{M}$ to no more than that of a second order, symmetric tensor as was done in [4]. This restriction, however, is only rigorously applicable to strain histories whose principal directions do not vary with time even though their principal values vary arbitrarily.

The restriction then implies that, for example, strain history damage effects can change an initially isotropic material into a transversely isotropic one and to, at most, an orthorhombic material.

1.5 Physical Consequences of $\mathcal{M}$

The use of the inequality in D1 and a few added mathematical assertions lends to the following set of results:

(1) Prior straining causes a change in the stress response moduli

(2) Damage produces compressive stresses in the unity configuration and an "elongation" type of permanent set in the stress-free configuration

(3) "Strain-hardening" and aging produce tensile stresses in the unity configuration and a "compressive" type of permanent set in the stress-free configuration

(4) Prior straining changes an initially isotropic material into an aelotropic material

(5) Changes in the structural component of a material cause "elastomorphic" behavior changes

(6) Changes in the viscous component of a material cause "viscomorphic" behavior

(7) A viscoelastic range can exist for most filled materials.

1.6 Summary of Work to Date

In order to provide an overview of the development of the ideas and methods herein presented, the next several pages constitute a summary of the effort with pertinent references.

Damage is defined as a modulus reduction caused by any microstructural weakening in a material arising from prior deformation histories (damage may be inherent in the material as a result of processing, then the "virgin" state may be inferred by a limit testing process but the "virgin" state is not accessible).

Consider two material samples to be compared for (tensorial) damage with S, the stress tensor; $\bar{n}$ an outward acting unit normal; D, the damage tensor; P, a deformation process, and G, the configuration in a relaxed state, we then say [7] the degree of damage $D_2 < D_1$ if and only if for every deformation process P starting from identical relaxed configurations G, the magnitudes of the projection of the surface tractions, $S_{\bar{n}}$, projected onto the exterior unit normal $\bar{n}$, are related by a dual isomorphism*

$$(S_1 \bar{n}) \cdot \bar{n} \leq (S_2 \bar{n}) \cdot \bar{n} \qquad (1\text{-}1)$$

with strict inequality holding in (1-1) for at least one P and one $\bar{n}$. Equality in (1-1) for every P, $\bar{n}$ means equality of damages $D_1 = D_2$. (Note: $D_2 < D_1 \rightarrow (S_1\bar{n}) \cdot \bar{n} \leq (S_2\bar{n}) \cdot \bar{n}$, i.e., D_1 leads to a "weaker" material, lesser modulus.)

* Note: the damage tensor is generally of second order for practical reasons and is in an inverse relation to $\mathcal{M}$.

1.6.1 Comparability of Damage

Since D is a symmetric tensor, not all D's may be comparable under the conditions of Equation (1-1).

We say that damage states obeying (1-1)are comparable. Otherwise, they are incomparable.

(Example: one-directional damage may have a weakening by (1-1) in that direction compared, to say, spherical (isotropic) damage in another specimen. But the spherical damage specimen will be weaker in a transverse direction, hence, the conditions of (1-1)are violated and the two damage states are incomparable, i.e., effectively consist of two different materials).

Thus we achieve a <u>partial</u> ordering on damage, D, so that our topology is a space of <u>chains</u>.

This implies (Zorn's Lemma) that a maximal element exists, i.e., failure when from (1-1)

$$(S\bar{n})\cdot\bar{n} \leq 0$$

(Either <u>no</u> stress can deform specimen, broken, or only compressive stresses can be sustained, i.e., "sticking" it back together.)

We postulate a virgin state which is the minimal element. Since (1-1)is a scalar number ordered, on a given chain, by damage D, we thus have a material ordering on D on a chain which can be normalized, say from 0 to 1 for comparable damages. [7]

<u>1.6.2 Semi-elastic Range (A Visco-elastic Range)</u>

So long as a material is subject to deformation processes in its semi-elastic (visco-elastic) range, no further damage ensues

and its relaxation modulus remains unchanged.

The term semi-elastic range is from [7] and visco-elastic range is defined and demonstrated in [8] and [9].

Note: For some materials a semi-elastic range may not exist, i.e., any deformation or motion may lead to increased damage.

1.6.3 Residual Stress

From [4] it is shown that if a damaged material is forced back into its unity (i.e., virgin) configuration, then residual compressive stresses will result. Proof: from (1-1) $S_1\bar{n}\cdot\bar{n} \leq S_2\bar{n}\cdot\bar{n}$ if $D_2 < D_1$. Consider material 2 as virgin. Then with both samples in original undistorted virgin configuration, the stress needed to maintain this configuration on the virgin sample is clearly zero, so

$$S_1\bar{n}\cdot\bar{n} \leq 0.$$

That is, any residual stresses on the damaged sample are zero or compressive.

1.6.4 Residual Stress Compressive Only

In [10] it is then proven that they cannot be zero, hence, the residual stress field on a damaged sample to force it into a (relaxed)

"virgin" configuration are somewhere compressive and possibly somewhere zero but are nowhere tensile.

1.6.5 Permanent Set

Postulating certain injectivity properties for the relaxed configuration defines various classes of materials [7 ,10]. These classes are shown in Figure 1, attached.

It is then proven in [10] that a damaged material must have permanent set in the relaxed, stress-free configuration. It is further proven that the permanent set configuration (3-D) is ordered by the partial ordering on damage, that is, given a chain of comparable damage states, the permanent set will be uniquely determined.

The inverse is, given a series of permanent sets caused by comparable damage states, the set uniquely determines the degree of damage.

1.6.6 Permanent Set "Extensile" in Nature

For a relaxed configuration it is shown in [12] that the stress is related to the deformation (strain measure) by a pseudo-linear relation

$$S = GC \tag{1-2}$$

with S, stress; G, a damage and "strain" dependent modulus (4th order tensor) and C, the strain measure. Equation (1-2) is the 1st order

approximation which can be made canonical (a la Rivlin-Spencer) as

$$S = G_0 1 + G_1 C + G_2 C^{-1}, \text{ etc.}$$

Regardless then, conditions of passivity or positive work done in a cycle, or linear thermodynamics, etc., cause G to be positive definite such that when the stress field is compressive somewhere and nowhere tensile (see 1.6.9 herein) then when it is removed, the material "expands" in a 3-D tensorial sense. Hence, since from all residual stresses are nowhere tensile, in a damaged material, removal of these stresses, i.e., letting them go to zero, leads from (1-2) to "expansion".

This fact with 1.6.6 herein then leads to the fact that the permanent set in a damaged material must be "expansive" in a 3-D sense (i.e., some directions may be shortened, i.e., transverse deformation in a simple tension test, but the overall field is "expansive").

1.6.7 Constitutive Equation

Quinlan and Fitzgerald [11] have shown that a first order expansion for the stress is given by

$$S = G_1 C + \int G_2(D,C,t-\tau)\,\dot{C}(\tau)d\tau + G_3 D + \int G_4(D,C,t-\tau)\dot{D}(\tau)d\tau \qquad (1\text{-}3)$$

where all kernels, G_1, G_2, G_3, G_4, are both damage and strain dependent in general. [11] rigorously legitimizes the second integral.

The presence of the fourth integral leads to an apparent "stiffening" of the modulus under active damaging, i.e., the material stress response shows a steepening with respect to strain when re-approaching the "virgin" curve as in the Mullin s' effect repeated loadings.

A computerized characterization method using (1-3) has been developed and is being used by United Technology Corporation on an investigative basis. The general approach is in [12,13].

Other characterizations such as those of Farris and Schapery are sub-classes of the above Equation (1-3) as shown in [14].

1.6.8 Thermo-mechanical Coupling

Lee [15] has derived an equation

$$S = G_1 C + \int G_2(D,\theta,C,t-\tau)\ \dot{C}(\tau)d\tau + \int G_3(D,\theta,C,t-\tau)\ \dot{D}(\tau)d\tau + \int G_4(D,\theta,C,t-\tau)\ \dot{\theta}(\tau)d\tau + G_5 D \qquad (1\text{-}4)$$

where θ is the temperature excursion from a rest state. For the case of $\dot{D} = 0$, he has found numerical values for G_4 and has successfully accounted for simultaneous cooling and straining results in damaged materials.

1.6.9 Chemical Aging

Hufferd, using a form of (1-4) but with H substituting for θ is using it with Thiokol (John Bennett) to account for chemical aging effects.

1.6.10 Hierarchy

The hierarchy of materials is given in Figure 1.

Semi-elastic means general linear or nonlinear visco-elastic.

Elastic means linear or nonlinear.

Mullins elastic covers elastic materials with Mullins effects and/or hysteresis.

All are sub-classes of my PMSE type, permanent memory with semi-elastic range.

(I adopt the word semi-elastic from Walter Noll, whose methods I have used to frame my PMSE concepts.)

This figure will appear in [10].

1.6.11 Metatropic Materials

Observing that prior damage producing straining processes effect not only a change in the value of the response functional but also produce a change in the isotropy group of the material, the term metatropic was coined by the author to describe these combined effects.

Metatropic Materials are defined as "materials which change or tend to change their properties in a specified manner (meta-) and which produce or tend to produce a more highly organized state (-tropic) in response to the sequence of past stimuli".

The above is presented in [16].

1.6.12 Finite Deformation Isotropy Groups

A parallel development has been the observation (rediscovery?) that materials may have higher order symmetries which appear isotropic under small strain. For example, a material which has transverse octagonal symmetry will exhibit identical response at 0°, 45°, 90°, at infinitesimal strain and thus will be isotropic.

However, at large strains, the octagonal symmetry will be observed. In general, for each higher polynomial in strain that is contained in the strain energy function for a material, two additional symmetry groups can be observed.

These results have application to fibre reinforced materials undergoing large strains. This work is covered in [17].

A loosely related paper evolving from the ideas developed herein is [18].

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STRUCTURE OF MATERIALS COVERED IN THIS REPORT

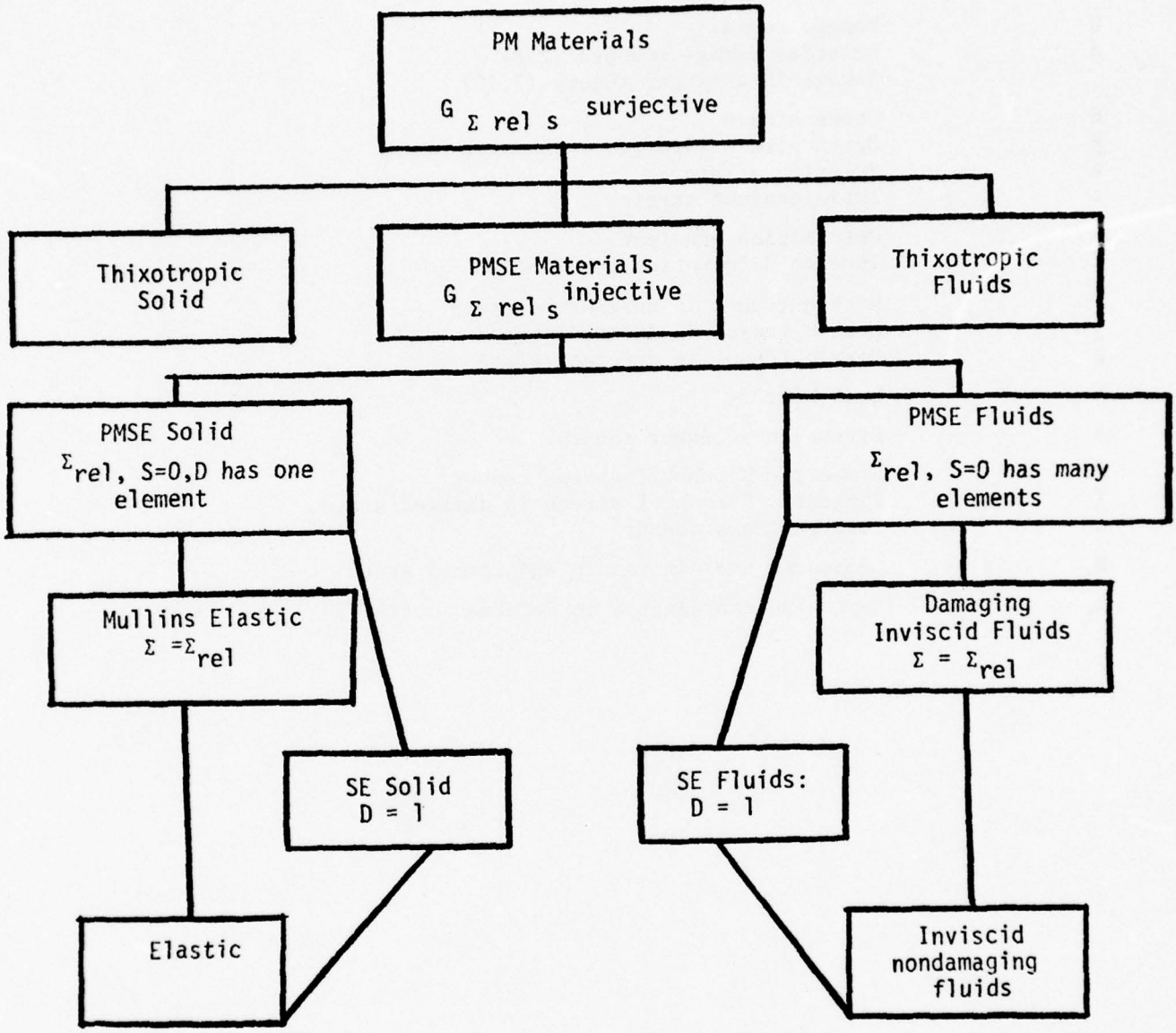

FIGURE 1

NOMENCLATURE, CHAPTER II

c, $c^{(0)}$, $c^{(1)}$	Modulus tensors
D	Damage tensor
d	Eulerian damage measure (2.8) Damage in a linear theory (2.10)
E	Green strain
$\underline{E}$	Green strain referred to damaged state
e	Eulerian strain
ε	Infinitesimal strain
F	Deformation gradient
f	Inverse deformation gradient
G	Metric tensor in undeformed state
$\underline{G}$	Metric tensor in damaged state
g	Metric tensor in deformed state
L	Load history
σ	Stress in a linear theory
T	Symmetric Kirchhoff stress tensor
$\underline{T}$	Symmetric Kirchhoff stress in damaged state
t	Cauchy stress tensor
X_i	Cartesian coordinates in undeformed state
x_i	Cartesian coordinates in deformed state

CHAPTER II: RECENT WORK

2.1 This report presents recent work on permanent memory materials. Permanent memory phenomena were first observed in uniaxial tests of highly filled solid propellants. When specimens were strained, brought to rest, and then retested, a reduction in moduli was observed. Moreover, an increase in the applied strain led to a larger modulus reduction. The term "permanent memory" was introduced to distinguish this behavior from the "fading memory" behavior of viscoelastic materials. The modulus reduction caused by strain is essentially irrecoverable; it is definitely not a transient phenomenon.

In the case of propellants, straining causes a degradation of material properties. Therefore, the change was called "damage". Subsequently, permanent memory properties were found to be fairly common; they do not always lead to degradation, but may result in improvement. Thus, by permanent memory we now mean any permanent change in properties due to strain history. The more specific term "damage" is retained here; nevertheless, it will be clear that our theory, suitably modified, can also model improvements in material properties.

To formulate the problem discussed here, suppose that a material is in some initial state in which its mechanical properties are known. Next, let it be subjected to arbitrary loads and then brought to rest; our problem is to predict the subsequent mechanical behavior. Because of permanent memory effects, it will not be the same as it was before loading.

Our treatment of this problem is based on certain key ideas from previous work. The first is the idea of a viscoelastic range. This term denotes strain histories to which a permanent memory material responds viscoelastically. Thus, in a viscoelastic range, no further change in material properties occurs. This idea simplifies our problem considerably. It now becomes a problem of relating viscoelastic characterizations before and after damage. Second, we use the suggestion of Fitzgerald [1] that damage be modeled by a second rank tensor. Our work actually sheds some light on when this model is valid. It leads to a definite mathematical problem which is ultimately solved by Rivlin's theory of tensor valued functions. Third, we use the experimental observation that damage is caused by tensile strains. From this idea, we can derive definite expressions relating the damage tensor to the strain history. In summary, our basic premises are:

(1) The existence of a viscoelastic range

(2) Damage is described by a second tank tensor.

(3) Damage results from tensile strains.

In section 2, the basic problem is formulated and discussed in greater detail. Section 4 introduces the tensorial damage measure. Damage dependent stress-strain relations are developed in sections 4 and 5. Section 6 describes the qualitative effects which this theory can model. In section 7, the stress-strain relations are rederived from Green's point of view by using a damage-dependent strain energy function. In section 8, the stress-strain relations are written in Eulerian variables; in section 9 they are written in terms of the damaged traction-free state. All of this work is valid for finite deformations.

In section 10, a linear theory for permanent memory materials is proposed.

The basic results are equations (5-2) and (10-1). Equation (5-2) is the finite damage dependent stress-strain relation for an initially isotropic material. The assumption and simplifications which lead to (5-2) are carefully stated in sections 2 through 5. Equation (10-1) is the corresponding formula for infinitesimal strains. According to this theory, the damaged material will contain permanent strains relative to the undamaged material. Also, the damaged material may be anisotropic. The minimum symmetry possible is orthorhombic, with crystal axes coincident with the principal axes of the damage tensor. Section 6 contains a detailed discussion of the symmetry properties of the damaged material.

Finally, we note that our goal has been to derive the most general equations consistent with our assumptions. This goal is in keeping with Rivlin's approach to deriving stress-strain relations. As a result, the theory is considerably more complex than necessary for materials such as propellants, for which experimental data contains a very high amount of scatter. However, since the theory is general, it can model many different qualitative properties. Once one determines which of these properties are appropriate for a material, characterization only requires choosing a few of the corresponding terms from (5-2), and curve fitting the experimental data.

2.2 Suppose that a permanent memory material has the stress-strain relation

$$t_{ij} = c^{(0)}_{ijkl}\, e_{kl} + c^{(0)}_{ijklmn}\, e_{kl}\, e_{mn} + \ldots \quad (*) \qquad (2\text{-}1)$$

This formula characterizes the material in the virgin state, or in any fixed reference state. The material is loaded arbitrarily and then brought to rest. Because of permanent memory effects, (2-1) no longer describes the material, and a new characterization is needed:

$$t_{ij} = c^{(1)}_{ij} + c^{(1)}_{ijkl}\, e_{kl} + c^{(1)}_{ijklmn}\, e_{kl}\, e_{mn} + \ldots \qquad (2\text{-}2)$$

In (2-2), the $c^{(1)}$ term reflects the possibility of permanent set: nonzero stresses may be needed to restore the original rest configuration. The problem of this section is to predict formula (2-2) knowing (2-1) and the intervening loads. Mathematically, the relationships

$$c^{(1)}_{ij} \ldots = c^{(1)}_{ij} \ldots (c^{(0)}_{ijke}, \ldots . ; L) \qquad (2\text{-}3)$$

are sought, where L stands for the damage producting loads.

Posing this problem is justified by the existence of a <u>viscoelastic range</u> in many permanent memory materials. Recall from [2] that this term denotes strain histories for which the material behaves viscoelastically. For example, in a uniaxial test if a permanent memory material is taken to a strain e_{max} and then brought to rest, subsequently the material is essentially viscoelastic as long as strains less than about

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* For notational simplicity, it will be understood that in formulas such as (2-1), the c's may be time dependent kernals, and the e's may be strain histories. Thus, the subsequent analysis will be valid for viscoelastic as well as for elastic behavior.

75% of e_{max} are applied. Thus, in (2-2) and (2-3) the e_{ij} are not arbitrary, but must be in a viscoelastic range.(*)

In many problems of practical interest, only equations (2-3) are needed; the transient formulas of Farris [3] and Quinlan [4] provide more information than necessary. One example is the thermal cycling problem: a rocket motor is subjected to a temperature loading and then brought to rest. The thermal loading produces damage; therefore, the material's subsequent mechanical behavior will be quite different from its original behavior. It is important to predict the subsequent behavior from the initial modulus and the thermal loads; that is, to determine relations (2-3). A transient description of the material properties during the thermal loading is unnecessary.

The relations (2-3) can be obtained as special cases of transient formulas such as Farris' or Quinlan's. However, (2-3) actually states the problem more generally because no restriction is placed on how the loads L enter the equations. Of course, this generality leads to an impractically complex theory, and simplifications must be sought. For example, consider the simplest special case of (2-3): the material is characterized in (2-1) and (2-2) by single fourth rank modulus tensors which are linearly related to each other. Then (2-3) becomes

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* If (2-1) is used to describe the virgin state, the viscoelastic range may reduce to infinitesimal strains. In this case, $c^{(0)}$ is the tangent modulus at zero strain. (2-1) cannot describe the hysteresis effects in the virgin state, but one can still pose the problem of relating the damaged moduli of (2-2) to the initial tangent modulus through formula (2-3).

$$c^{(1)}_{ijkl} = D_{ijklmnop}(L)\, c^{(0)}_{mnop} \qquad (2\text{-}4)$$

where an eighth rank tensor function D of the loads L must be determined experimentally. Even allowing for the index symmetries in $c^{(0)}$ and in $c^{(1)}$, (2-4) requires the evaluation of an excessive number of constants. Furthermore, the dependence of D on L is still unspecified.

In [1] Fitzgerald suggested that damage could be measured by a second rank tensor. This idea is motivated by observing that damage could reduce a material's modulus by different amounts in different directions. Thus, a damaged initially isotropic material could well be anisotropic. A second rank tensor is the simplest quantity which can model this possibility. Of course, more complex quantities could also be used: formula (2-4) illustrates the use of an eighth rank tensor to model damage. In any case, in general damage cannot be described simply by a scalar.

Analytically, Fitzgerald's suggestion is incorporated into equation (2-3) by replacing L by a second rank tensor D_{ij} which depends on L:

$$c^{(1)}_{ij\ \ldots} = c^{(1)}_{ij\ \ldots}\ (c^{(0)}_{ijkl,\ldots}\ ;\ D_{ij})\ (*) \qquad (2\text{-}5)$$

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* Equation (2-5) should be normalized so that the moduli are unchanged if there is no damage. For later work, it will be convenient to let $D_{ij} = 0$ corresond to no damage; then

$$c^{(1)}_{ij\ \ldots}\ (c^{(0)}_{ijke,\ldots}\ ,\ 0) = c^{(0)}_{ij\ \ldots}$$

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The relation between D and L is still unknown, however, the general expansion (2-4) is easily worked out. The details are carried out in sections 4 and 5, and some consequences of (2-4) are discussed there. It should be emphasized that assuming the loads L enter (2-3) through a second rank tensor is a simplification which may not be valid under all conditions. The character of the loads L may be important in determining the validity of the approximation. This issue is discussed further in section 3.

2-3 Existing experimental evidence does not suggest how L enters equation (2-3) when L is a completely arbitrary loading history. The reason is that only a special class of loading histories is usually produced in the laboratory, namely those histories with principal axes of stress and strain constant in time. Uniaxial tests, the most commonly performed on propellant, are obviously of this type. Assuming that damage is produced by this kind of loads is not necessarily too restrictive since practical problems, such as the thermal cycling problem, often involve precisely such conditions.

Keeping in mind the restriction to constant principal directions, experiments do suggest that

(a) Damage is caused by tensile strains.

(b) The maximum tensile strain is the most important factor is causing damage.

Thus, as a first approximation, D_{ij} may be taken as the maximum tensile strain during the loads L. A suitable measure of this strain will now be developed. Denote the strain history by E(t), and let E(t) be reduced to principal axes by the time-independent rotation matrix R. Then

$$\begin{bmatrix} E_{xx}(t) & 0 & 0 \\ 0 & E_{yy}(t) & 0 \\ 0 & 0 & E_{zz}(t) \end{bmatrix} = R^T E(t) R$$

where x, y, and z denote principal axes.(*) Let E^* be the maximum value of E(t). Then

$$D = -R \begin{bmatrix} E^*_{xx} - 1 + |E^*_{xx} - 1| & & \\ & E^*_{yy} - 1 + |E^*_{yy} - 1| & \\ & & E^*_{zz} - 1 + |E^*_{zz} - 1| \end{bmatrix} R^T/2 \qquad (3\text{-}1)$$

is a measure of maximum tensile strain which vanishes if the principal strains are never tensile. Alternatively, damage might be measured by some function f(D). If, for example, tensile strains below a certain threshold E_T did not cause damage, one might choose instead of (3-1),

$$-D = \frac{1}{2}R \begin{bmatrix} E^*_{xx} - E_T + |E^*_{xx} - E_T| & & \\ & E^*_{yy} - E_T + |E^*_{yy} - E_T| & \\ & & E^*_{zz} - E_T + |E^*_{zz} - E_T| \end{bmatrix} R^T$$

Once D is related to L, the relations (2-3) between the initial and damaged moduli are obtained by using the results of sections 4 and 5.

One problem which does not involve loading histories with constant principal directions can actually be handled by (2-3). Suppose that the

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* Recall that $E_{ij} = \delta_{ij}$ in the unstrained state and that principle strains $E > 1$ are tensile, while principal strains $E < 1$ are compressive.

virgin sample with moduli $c^{(0)}$ is subjected to damage D^0 and brought to rest. New moduli $c^{(1)}$ result. Now let the material be subjected to another damage field D^1 with different principal axes from D^0. The resulting moduli $c^{(2)}$ are expressed in terms of the $c^{(1)}$ by (2-3):

$$c^{(2)}_{ij\ \ldots} = c^{(2)}_{ij\ \ldots}\ (c^{(1)}_{ij,'\ldots.},\ D^1)$$

To express the $c^{(2)}$ in terms of the $c^{(0)}$, use (2-3) to express the $c^{(1)}$ in terms of $c^{(0)}$. The result is

$$c^{(2)}_{ij\ \ldots} = c^{(2)}_{ij\ \ldots}\ (c^{(1)}_{ij\ \ldots}\ (c^{(0)}_{ijkl,\ \ldots.};\ D^0),\ \ldots\ ,\ D^1)$$

Note that D^0 and D^1 enter this equation in a complex way. There need not be any one second rank tensor D^2 such that

$$c^{(2)}_{ij}\ \ldots = c^{(2)}_{ij}\ \ldots\ (c^{(0)}_{ijke},\ \ldots\ ,\ D^2)$$

This example indicates the complexities possible if damage is produced by general loading histories. This issue is not pursued any further here since its practical significance is unclear.

2-4 The damage-dependent stress-strain relations can be derived directly from equation (2-5). However, it is simpler to proceed by substituting (2-5) into (2-2). The resulting expression for stress depends on both strain and damage:

$$T = T(D,\ E) \tag{4-1}$$

Equation (4-1) is really equivalent to (2-5); indeed, if D is treated as a parameter in (4-1), the moduli of various orders can be derived, and the relations (2-5) recovered. The difference between the two equations is

only the point of view. In (4-1), only D and E are variable, whereas (2-5) allows the moduli $c^{(0)}$ to vary as well. Thus, the form (2-5) is required only when the moduli are changing; typical cases are superposing damage fields as in section 3, or developing a transient theory.

The easiest approach to simplifying (4-1) is based on polynomial approximation. If the D_{ij} and the E_{ij} do not exceed a fixed maximum, Weierstress' theorem implies that the T_{ij} may be taken as polynomials in the D_{ij} and the E_{ij} and the difference between the polynomial form and the hypothetical exact form may be made arbitrarily small. Thus, one can set

$$T_{ij} = \sum_{p,q} a^{(p,q)}_{ijklmn} (E_{kl})^p (D_{mn})^q \qquad (4\text{-}2)$$

The precise range on p and q in (4-2) is not specified, but it is assumed to be finite. Equation (4-2) is still too general since the components T_{ij} that it defines may not transform as a second rank tensor. To assure the proper transformation laws T must be built up from D and E by tensor operations, namely by tensor products and by contractions with constant tensors. Invoking these principles, (4-2) becomes

$$T_{ij} = T^{(00)}_{ij} + T^{(10)}_{ijkl} D_{kl} + T^{(01)}_{ijkl} E_{kl} + T^{(20)}_{ijklmn} D_{kl} E_{mn} + \ldots \qquad (4\text{-}3)$$

where $T^{(ij)}$ is a constant $2(i+j+1)^{th}$ rank tensor. Alternatively, (4-3) may be rewritten by collecting like products of E. Then

$$T_{ij} = T^{(0)}_{ij}(D) + T^{(1)}_{ijkl}(D) E_{kl} + T^{(2)}_{ijklmn}(D) E_{kl} E_{mn} \qquad (4\text{-}4)$$

where

$$T_{ij}^{(0)} = T_{ij}^{(00)} + T_{ijkl}^{(10)} D_{kl} + T_{ijklmn}^{(20)} D_{kl} D_{mn} + \ldots \qquad (4\text{-}5)$$

$$T_{ij}^{(1)} = T_{ij}^{(01)} + T_{ijkl}^{(11)} D_{kl} + T_{ijklmn}^{(21)} D_{kl} D_{mn} + \ldots$$

It should be emphasized that all of the sums in (4-3), (4-4), and (4-5) are finite. Of course, in practice, only a very small number of terms would be taken.

Equation (4-4) can be used to obtain approximate theories of various orders. Consider first the case of infinitesimal strain. Then all products involving E can be ignored, and then (4-4) becomes, writing ε for E:

$$\sigma = \sigma^{(0)}(D) + c^{(1)}(D)\,\varepsilon \qquad (4\text{-}6)$$

In (4-6), $\sigma^{(0)}$ is the residual stress present in the zero strain configuration, and $c^{(1)}(D)$ is the damage dependent modulus tensor.

Next consider the case that D is small and the strains are roughly an order of magnitude smaller than D. This case would apply to a test sequence in which a material is subjected to strains of around 10% and subsequently tested at strains of one or two percent. Then it is reasonable to retain terms of order D, D^2, and D^3 and DE in (4-6) and to discard the others; thus, $\sigma^{(0)}(D)$ is at most cubic in D and $c^{(1)}$ is linear in D. In view of the difficulty in measuring the residual stresses accurately, it is not unreasonable to take $\sigma^{(0)}$ linear in D as well. This procedure leads to the simplest theory exhibiting permanent stresses and modulus reduction:

$$\sigma_{ij} = \sigma^{(0)}_{ij} + c^{(1)}_{ijkl}\,\varepsilon_{kl};\ \sigma^{(0)} \text{ and } c^{(1)} \text{ are linear in D.} \qquad (4\text{-}7)$$

Many test conditions closely approximate the requirements for (4-7). This fact, together with its simplicity, makes (4-7) a practical approximate theory.

2.5 The tensor transformation of section 3 does not impose any restriction on the tensors $T^{(ij)}$ appearing in (4-3). The form of these tensors can only be determined from considerations of material symmetry. In this section, the stress-strain relations for an initially isotropic material will be developed. For such a material, the tensors $c^{(0)}_{ij}$ ... of (2-1) are isotropic; therefore, they are sums of tensor products of Kronecker deltas. It seems reasonable that the $T^{(ij)}$ will also be isotropic tensors. If so, the material will be said to exhibit <u>isotropic damaging</u>.(*) Thus, if strain and damage are rotated arbitrarily while the material is fixed, the stress is rotated the same way. On the other hand, it is not impossible for an initially isotropic material to damage anisotropically. For example, a given tension might produce greater degrees of weakening in some directions than in others. Whether or not the $T^{(ij)}$ have the same symmetry properties as the $c^{(0)}$ can only be determined by experiments.

The general form of equation (2-2) for an initially isotropic material with isotropic damaging is found by replacing the $T^{(ij)}$ in (4-3) by arbitrary products of Kronecker deltas. The expression that results is clearly a sum of terms of the form $\mathrm{tr}(D^m E^n)\, D^p\, E^q$. Rivlin

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* Analytically,

$$R^T T R = T(R^T D R,\ R^T E R) \qquad (5\text{-}1)$$

has shown (Green and Adkins [5], appendix) that this sum can be reduced to the form

$$T = f_o I + f_1 E + f_2 E^2 + f_3 D + f_4 D^2 + f_5 (ED + DE) + f_6 (ED^2 + D^2 E) + f_7 (E^2 D + DE^2) + f_8 (E^2 D^2 + D^2 E^2) \tag{5-2}$$

where the f_i are polynomials in the invariants

$$\text{tr}E,\ \text{tr}E^2,\ \text{tr}E^3,\ \text{tr}D,\ \text{tr}D^2,\ \text{tr}D^3,\ \text{tr}ED,\ \text{tr}E^2D,\ \text{tr}ED^2,\ \text{tr}E^2D^2 \tag{5-3}$$

Let D be fixed in (5-2). The resulting stress-strain relation has orthorhombic symmetry with crystal axes coincident with the principal axes of D. To verify this fact, recall that (5-2) satisfies (5-1). Thus,

$$R^T T R = T(D,\ R^T E R)$$

holds whenever $R^T D R = D$. Therefore, (5-2) with D fixed is invariant under symmetry operations which leave D invariant. A reflection through any plane formed by D's principal axes leaves D unchanged, and hence leaves (5-2) unchanged.(*) Any other symmetry operation will in general change the form of D and hence change the form of (5-2). Higher degrees of symmetry are possible only if D is more symmetric. Thus, if D has two equal principal values, it is invariant under all rotations in the plane formed by the corresponding principal axes. D, and therefore (5-2)

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* These are the symmetries characteristic of an orthorhombic material.

are then transversely isotropic. Finally, if D has three equal principal values, it is invariant under all rotations; (5-2) is then fully isotropic.

Thus, imposing a single damage field transforms an isotropic material into an orthorhombic material. Note, however that the orthorhombic symmetry represented by (5-2) is of a special type. For instance, cubic symmetry cannot occur in the form (5-2). Interchanging the principal axes of D leaves D invariant only if D has three equal principal values; in this case, (5-2) is fully isotropic. This conclusion also results from a count of arbitrary constants. The general linear orthorhombic material is characterized by nine constants. In (5-2), however, the arbitrary constants in a linear theory are the two initial isotropic moduli, and the three principal values of D, for a total of only <u>five</u> constants.

2.6 Equation (5-2) contains eight arbitrary polynomials $f_1 \ldots f_8$. Therefore, theories containing any number of free constants are available for material characterization in the damaged regime. Once a definite theory has been selected, its unknown constants are evaluated by curve fitting to experimental data on laboratory tests such as uniaxial tension. This procedure is no different in principle from that used in finite elasticity (see for example, [5], ch. 11). Just as in finite elasticity, care must be taken to select a <u>consistent</u> theory. For example, if a quadratic term in E appears in the theory, then all possible quadratic terms in E must appear as well. Tests may reveal that some of these additional terms have zero coefficients, but this situation must be established by experiments and should not be assumed a priori.

The use of (5-2) in material characterization is simplified somewhat since its terms correspond to a few easily identified qualitative effects in the damaged state. Accordingly, (5-2) will now be analyzed in more detail from this point of view. In what follows, p(D) and q(E) will denote polynomials in the scalar invariants of D and E, *not* necessarily the same ones each time they appear. Also, it will be convenient to write the stress-strain relations in terms of a strain measure, for example E - I, which vanishes in the undamaged unstrained state.

(a) Terms independent of E: $p(trD, trD^2, trD^3)I$, $p(trD, trD^2, trD^3)D$, $p(trD, trD^2, trD^3)D^2$

Let the damaged material be brought to its original unstrained state. These terms then appear in (5-2) as stresses necessary to do this. Measurements of these stresses would thus be used to evaluate coefficients which correspond to the terms (a). Since the stress T is measured from the initial rest configuration, p I is a purely hydrostatic stress, while the other two terms contribute stresses which depend on the principal directions of D. Evaluating the polynomials p(D) will, of course, require experiments with different degrees of damage D.

Terms (a) may be regarded alternatively as measures of <u>permanent set</u>: this is the strain present in the damaged configuration under zero stress.(*) To evaluate this strain, let T = 0 in (5-2) and solve for

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* More precisely, "strain" means strain measured with respect to the initial rest state, and by "zero stress," zero surface traction is understood.

E. Because of terms (a), E = 0 is not in general the solution. Theoretically, this interpretation makes it possible to evaluate coefficients corresponding to (a) by measuring permanent set, rather than the stresses needed to remove it. However, this process requires the solution of (5-2) with T = 0; in effect, (5-2) must be solved for E in terms of T and D. This solution will be practical only in very simple cases.

(b) $$p(D)q(E)\ E,\ p(D)q(E)\ E^2$$

D enters these terms through its scalar invariants only. The principal directions of D therefore play no role. If just terms (b) appear in (5-2), an initially isotropic material remains isotropic after damage. Thus, if damage was caused, for example, by a uniaxial tension, subsequent uniaxial tests would give identical results in all directions of test. The direction of the damage field would not be distinguished. This situation can be called <u>isotropic damage</u>.

Qualitatively, a number of different effects can be modeled with these terms. Let the initial stress-strain relation be $T = T(E) = q_o(E)I + q_1(E)E + q_2(E)E^2$. If subsequent to damage, $T = [1 + p(D)]T(E)$, all stress-strain curves are reduced proportionally; that is, the damaged curve for any test is a constant multiple of the undamaged curve. Furthermore, the constant of proportionality is the same for all tests, and depends only on the degree of damage. This is the simplest theory of isotropic damage.

A more general theory results from setting

$$T = q_o(E)[1 + p_o(D)]I + q_2(E)[1 + p_2(D)]E + q_2(E)[1 + p_2(D)]\,E^2$$

In this case, the damaged and undamaged curves for the same test need not be proportional. Finally, if arbitrary terms (b) appear, different cross-effects can be present in the damaged and undamaged states.

(c) $\quad q(\det E)p(D)I,\ q(\det E)p(D)D,\ q(\det E)p(D)D^2$

First, note that since det E is a polynomial invariant of E, it can be written as a polynomial in tr E, tr E^2, and tr E^3. When q(1) = 0, these terms have the following interpretation. Let the damaged material be returned to its initial unstrained state. Any stresses needed to do this are ignored in what follows. These stresses would be covered under (a) in any case. Now subject the damaged sample and an undamaged sample to identical volume-preserving deformations.(*) Since then det E = 1, and q(1) = 0, when only terms (c) appear in (5.2), the stresses in the undamaged and damaged bodies are the same. Thus, there is no difference in mechanical response to a volume preserving deformation.

When a deformation which is not volume preserving is applied, terms (c) represent additional stresses required to impose the deformation in the damaged state. The first term in (c) is distinguished from the other two by its behavior under volumetric strain tests. If such a test requires just hydrostatic stress, then only the first term is present in

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* "Volume preserving" means volume-preserving with respect to the undamaged, unstrained state.

the stress-strain relation. If the stress is not hydrostatic, then the other two terms are present. Note that in this case, the principal axes of stress must coincide with the principal axes of D.

These terms are not appropriate for materials which are incompressible in both the damaged and undamaged state, since then det E is constant.(*)

When $q(1) \neq 0$ or when det E is replaced by other invariants of E, then terms (c) have analogous, but physically somewhat implausible interpretations. Thus, if $q(a) = 0$, the damaged and undamaged materials respond identically to deformations such that det E = a. If also, det E is replaced by some invariants $I_i(E)$, and $q(a_1, \ldots a_n) = 0$, then the mechanical response is identical when $I_i(E) = a_i$. If q is never zero, terms (c) simply represent additional stresses needed to impose certain deformations on the damaged state; these stresses depend only on certain invariants of the deformation. These terms would seem to be of most interest in the case discussed at first.

Finally, to obtain the corresponding terms in a linear theory with infinitesimal strains, det E is replaced by tr E, and necessarily $q(x) = Ax$, otherwise the theory is not linear. Then terms (c) have the additional interpretation that the damaged and undamaged shear modulus are the same.

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* Of course, the constant need not be the same in both the undamaged and damaged states since damage may cause an increase in volume.

(d) Terms containing products of E and D and their invariants.

These terms cause anisotropic behavior in the damaged state. For example, let $D_{xx} = D$, all other $D_{ij} = 0$; this damage field would be caused by extension in the x direction. Now let E represent an arbitrary uniaxial tension. Then trDE has a maximum for tension in the x direction and will have minimums for tension in the y and z directions. This fact makes it easy to determine whether or not tr DE should appear in the damaged stress-strain relation. The other joint invariants of E and D: tr E^2D, tr ED^2, and tr E^2D^2, have similar properties and can be used for curve fitting.

A simple example of (d) is $T = (1 + a \text{ tr } DE)\, T(E)$ where, as in (c), $T = T(E)$ is the undamaged stress-strain relation. In this theory, the stress-strain curves in the damaged and undamaged material are proportional for any test procedure. However, the constant of proportionality depends on the relative orientation of E and D. As in (c), more complex formulas of this type can be constructed quite easily. Thus, in

$$T = q_o(E)[1 + p_o(D)\ ] + q_1(E)[1 + p_1(D)]E + q_2(E)[1 + p_2(D)]E^2$$

the damaged and undamaged curves are not proportional, and the degree to which they differ depends on the orientation of D and E.

The terms ED + DE, etc. act similarly when only tension tests are involved. They can be distinguished from the terms involving joint invariants only by their behavior under volumetric straining of the damaged material. Let E = a I. Then the joint invariants are multiples

of the invariants of D. When only these terms occur in (5-2), the principal directions of D have no effect on the damaged material, and a volumetric strain requires only hydrostatic stress. But if the terms ED + DE etc. are present, the volumetric strain will require a non-hydrostatic stress.

For incompressible materials, the volumetric strain is inadmissable. Then the two terms can be distinguished by biaxial damage fields. Let

$$D = \begin{bmatrix} D_{xx} & & \\ & D_{yy} & \\ & & 0 \end{bmatrix} \quad E = \begin{bmatrix} E_{xx} & & \\ & E_{yy} & \\ & & E_{zz} \end{bmatrix}$$

When only ED + DE etc. occur in (5-2), the stresses in the x and y directions are independent, whereas the joint invariants introduce coupling.

(a) through (d) exhaust all of the terms which can occur in (5-2). Summarizing their properties, terms (a) are required whenever permanent set is observed. When damage is isotropic, terms (b) are sufficient for a characterization. Isotropic damage is easily detected from tension tests in different directions. The description of anisotropic damage requires terms (d). Finally, (c) describes specialized situations when the damaged and undamaged materials respond identically to certain types of deformation.

2.7 The derivation of stress-strain relations in sections 4 and 5 follows Cauchy's approach of relating the stress and strain tensors directly. The alternative approach of Green, in which the relations are obtained through a strain energy function, will be outlined in this

section. This formulation is especially useful in a coupled thermo-mechanical analysis since then the strain energy function need only be replaced by the Helmholtz function.

Let the initial strain energy function be $W_o(E_{ij})$.(*) Then the initial stress-strain relation is

$$T_{ij} = \rho_o \frac{\partial W_o}{\partial E_{ij}}$$

After a damage-producing load sequence L, the strain energy is $W(E_{ij}, L)$. As in section 4, Fitzgerald's suggestion that damage be measured by a second rank tensor D_{ij} leads to the strain energy $W(E_{ij}, D_{ij})$. The relation between D_{ij} and L has already been discussed in section 3. As in section 4, when the components E_{ij} and D_{ij} do not exceed a fixed maximum, Weierstrass' theorem may be invoked to approximate W by a polynomial in the E_{ij} and D_{ij}. No further restriction on W is possible without bringing in material symmetry properties. If a material exhibits isotropic damaging (see section 5), W is then an isotropic function of E and D. Then Rivlin's reduction (see [5], p. 317) leads to the conclusion that W is a polynomial in the basic invariants

$$\begin{array}{llll} I_1 = \text{tr } E & I_2 = \text{tr } E^2 & I_3 = \text{tr } E^3 & \\ J_1 = \text{tr } D & J_2 = \text{tr } D^2 & J_3 = \text{tr } D^3 & \\ K_1 = \text{tr } DE & K_2 = \text{tr } E^2D & K_3 = \text{tr } ED^2 & K_4 = \text{tr } D^2E^2 \end{array} \qquad (7\text{-}1)$$

---

* W_o may be the potential for the initial tangent modulus at zero strain. See section 3.

The damage dependent stress-strain relation is

$$T_{ij} = \frac{\partial W}{\partial I_1}\frac{\partial I_1}{\partial E_{ij}} + \ldots + \frac{\partial W}{\partial K_4}\frac{\partial K_4}{\partial E_{ij}} \tag{7-2}$$

The partial derivatives of the invariants with respect to E_{ij} are as follows:

$$\frac{\partial I_1}{\partial E_{ij}} = \frac{\partial E_{kk}}{\partial E_{ij}} = \delta_{ij}$$

$$\frac{\partial I_2}{\partial E_{ij}} = \frac{\partial}{\partial E_{ij}} E_{kl}\, E_{lk} = 2\, E_{ij}$$

$$\frac{\partial I_3}{\partial E_{ij}} = \frac{\partial}{\partial E_{ij}} E_{kl}\, E_{lm}\, E_{mk} = 3E_{im}\, E_{mj} = 3(E^2)_{ij}$$

$$\frac{\partial J_k}{\partial E_{ij}} = 0$$

$$\frac{\partial K_1}{\partial E_{ij}} = \frac{\partial}{\partial E_{ij}} D_{kl}\, E_{lk} = D_{kl}\, \delta_{il}\, \delta_{jk} = D_{ij}$$

$$\frac{\partial K_2}{\partial E_{ij}} = \frac{\partial}{\partial E_{ij}} E_{kl}\, E_{lm}\, D_{mk} = E_{im}\, D_{mj} + D_{im}\, E_{mj} = (ED + DE)_{ij}$$

$$\frac{\partial K_3}{\partial E_{ij}} = (D^2)_{ij}$$

$$\frac{\partial K_4}{\partial E_{ij}} = (ED^2 + D^2E)_{ij}$$

Substituting into (7.2),

$$T_{ij} = \frac{\partial W}{\partial I_1}\delta_{ij} + 2\frac{\partial W}{\partial I_2}E_{ij} + 3\frac{\partial W}{\partial I_3}(E^2)_{ij}$$

$$+ \frac{\partial W}{\partial k_1}D_{ij} + \frac{\partial W}{\partial k_2}(ED + DE)_{ij} + \frac{\partial W}{\partial k_3}(D^2)_{ij} + \frac{\partial q}{\partial k_4}(ED^2 + D^2E)_{ij}$$

As expected, this is a special case of formula (5.2).

2.8 The stress-strain relations have been developed using the undeformed body as the reference configuration; this is the Lagrangian formulation.(*) In this section, the alternative formulation using the deformed configuration as the reference (the Eulerian formulation) is developed.

In finite elasticity, the choice between the Eulerian and Lagrangian viewpoints is basically arbitrary. In the theory of permanent memory materials, however, the Lagrangian viewpoint has the definite advantage that the damage tensor D, defined in the undeformed undamaged body, is constant. When the deformed body is the reference configuration, the damage measure depends on the current configuration. For this reason, the Lagrangian formulation will be taken as the basic formulation, and the Eulerian equations will be derived formally from it.

Recall that the basic equations for isotropic damaging were:

$$T = f_1 I + f_2 D + f_3 E + f_4 D^2 + f_5 E^2 + f_6(ED + DE)$$

$$+ f_7(ED^2 + D^2E) + f_8(E^2D + DE^2) + f_9(E^2D^2 + D^2E^2) \qquad (8.1)$$

---

* More precisely, the undamaged undeformed configuration is the reference.

where the f_i are polynomials in the invariants (5-3). T is the symmetric Kirchoff stress, and E is the Green strain defined in Cartesian coordinates by $ds^2 = E_{ij}\, dX_i dX_j$. The Eulerian equations use the Cauchy stress t and the strain e defined by $dS^2 = e_{ij}\, dx_i dx_j$. Before transforming to Eulerian variables, it is important to decide whether (8-1) contains contravariant, covariant, or mixed tensors. Ordinarily, this distinction is not important since the tensor types coincide when only Cartesian coordinates are used. However, making a transition from Lagrangian to Eulerian variables generally involves transformation to an oblique curvilinear coordinate system, even if Cartesian coordinates are used in the reference configuration. Thus, the distinction is important in this context.

The simplest formulas result if (8-1) is written in terms of mixed tensors. Since a self-contained comprehensible discussion of tensor analysis is impractical we list the important properties of mixed tensors and refer to [6] for details. Mixed components of a covariant tensor A_{ij} are defined by

$$A^i_j = G^{ik} A_{kj} \tag{8-2}$$

where $dS^2 = G_{ij}\, dX^i\, dX^j$, and $G^{ik} G_{kj} = \delta^i_j$. If $x_i = x_i\,(X_j)$ defines a change of variables, A^i_j transforms as

$$A^{i\prime}_j = \frac{\partial x_i}{\partial X_k} \frac{\partial X_l}{\partial x_j} A^k_l \tag{8-3}$$

Denote by X_i and x_i Cartesian coordinates in the undeformed and deformed bodies respectively. In terms of the deformation gradients

$$F^i_j = \frac{\partial x_i}{\partial X_j} \qquad f^i_j = \frac{\partial X_i}{\partial x_k}$$

(8-3) becomes

$$A'^{i}_{j} = f^i_k \, f^l_j \, A^k_l$$

or in matrix notation

$$A' = FAf \tag{8-4}$$

Since, in view of the chain rule $F^i_k \, f^k_j = f^i_k \, F^k_j = \delta^i_j$, $f = F^{-1}$. Thus, the mixed components transform like a matrix under a change of variables.(*) Thus, mixed components are particularly suitable for matrix equations such as (8-1). Indeed, expressions such as

$$(E^2)^i_j = E^i_k \, E^k_j, \quad (ED)^i_j = E^i_k \, E^k_j, \quad trE = E^k_k$$

are valid tensor operations, while corresponding covariant or contravariant expressions are more complex.

The mixed Cauchy stress tensor t^i_j is related to T^i_j by

$$Jt^i_j = F^i_k \, f^l_j \, T^k_l \qquad (JT = FTf)$$

where $J = \det F$. Define the Eulerian damage measure d^i_j by

$$d^i_j = F^i_k \, f^l_j \, D^k_l \qquad (d = FDf)$$

As expected, the Eulerian damage depends on the current configuration. Next, note that

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* This fact makes it particularly convenient to define the mixed damage tensor D^i_j in terms of the E^i_j. Recall from section 3 that D was defined as a matrix function of the strain history E.

$$FEf = FF^{T}Ff = FF^{T} = e^{-1}$$

Therefore, if p(D,E) denotes any polynomial in D and E appearing in (8-1),

$$Fp(D, E)f = p(d, e^{-1}) \tag{8-6}$$

It follows from this equation that the joint invariants transform according to

$$\text{tr } p(D, E) = \text{tr } Fp(D, E)f = \text{tr } p(d, e^{-1}) \tag{8-7}$$

Substituting (8-5) - (8-7) into (8-1),

$$\begin{aligned} Jt^{i}_{j} &= g_1 \delta^{i}_{j} + g_2 d^{i}_{j} + g_3 e^{+i}_{j} + g_4 (d^2)^{i}_{j} + g_5 (e^{-2})^{i}_{j} \\ &+ g_6 (e^{-1}d + de^{-1})^{i}_{j} + g_7 (e^{-1}d^2 + d^2e^{-1})^{i}_{j} + g_8 (e^{-2}d + de^{-2})^{i}_{j} + g_9 (e^{-2}d^2 + d^2e^{-2}) \end{aligned} \tag{8-8}$$

where the g_i result from replacing E by e^{-1} and D by d in the f_i.

Eulerian stress-strain relations with covariant or contravariant components are obtained from (8-8) by raising or lowering indices with the metric tensors g^{ij} and g_{ij}. But it is important to bear in mind that since (8-8) is a transformation of (8-1), these are the metrics induced in the deformed body by the deformation x_i (X_j) when the undeformed body has the usual Euclidean metric. Thus,

$$g^{ij} = \sum_{1 \le a \le 3} F^{i}_{a} F^{j}_{a} = (e^{-1})^{ij}; \quad g_{ij} = \sum_{1 \le a \le 3} f^{a}_{i} f^{a}_{j} = e_{ij}$$

This situation causes the contravariant and covariant Eulerian equations

to be more complicated than (8-1). Since these expressions will not be used in what follows, rather than working them out in detail, we just cite one typical case: suppose that one wants to raise indices in $(d^2)^i_j$. Multiplying by g^{lj},

$$d^i_k \, d^k_l \, g^{lj} = d^i_k \, d^{kj} = d^{il} \, g_{lk} \, d^{kj} = (ded)^{ij}$$

Thus, in the contravariant form of (8-8), d^2 is replaced by ded, and tr d^2 by tr ded.

2.9 So far, the finite stress-strain relations have been expressed in terms of the original undeformed state, and in terms of the current deformed state. With permanent memory materials, it is also possible to state the relations in terms of the damaged traction-free state.

This alternative does not lead to a universal form of the stress-strain relations, as do the Eulerian and Lagrangian equations. The reason is that the damaged traction free state depends on the geometry of the body and on the damage field. In view of this fact, we will only outline the procedure for obtaining these equations.

First, it is necessary to determine the damaged traction free state. This requires solving the equilibrium equations

$$\frac{\partial}{\partial X_i} T_{ij} \, F^k_j = 0 \qquad (9\text{-}1)$$

strain-deformation relations

$$E_{ij} = \frac{\partial x_i}{\partial X_a} \frac{\partial x_j}{\partial X_a} \qquad (9\text{-}2)$$

and stress-strain relations (5-2) subject to the boundary condition of zero surface traction. Because of the terms independent of E in (5-2), the solution need not be E = I, T = 0. Denote the solution for the deformations by x_i and the corresponding deformation gradient by F. The problem is to reformulate (5-2) in terms of the new configuration defined by the x_i. This problem is analogous to the transition to Eulerian variables of section 8; the only difference is that F is a constant deformation gradient field, so that the strains transform differently. Denote the mixed components of stress, strain, and damage in the traction free state by $\underline{T}^i_j$, $\underline{E}^i_j$, and $\underline{D}^i_j$. Then

$$\begin{aligned} \underline{T}^i_j &= F^i_k \, f^j_1 \, T^k_1 \\ \underline{E}^i_j &= F^i_k \, f^j_1 \, E^j_1 \\ \underline{D}^i_j &= F^i_k \, f^j_1 \, D^k_1 \end{aligned} \tag{9-3}$$

Substituting into (5-2),

$$\underline{T}^i_j = T^i_j \, (\underline{D}^i_j, \, \underline{E}^k_j) \tag{9-4}$$

Recall that coordinates have been so chosen that $\underline{T}_n = 0$ on the boundary of the reference configuration when $\underline{E} = I$.

To raise or lower indices in (9-4), the metric tensors $\underline{G}_{ij}$ and $\underline{G}^{ij}$ must be used. As in section 8, these are the metrics induced in the damaged state by the deformation x_i when the undamaged state has the Euclidean metric $dS^2 = dX^2 + dY^2 + dZ^2$. Thus, (9-4) is based on a coordinate system which is in general oblique and curvilinear.

It is also possible to obtain analogs of (9-4) in which $\underline{G}^{ij}$ and $\underline{G}_{ij}$ have a simpler form. The optimal form for the metric will depend on the geometry of the damaged state; suppose, for example, that it is desired to use the Euclidean metric. Then the metric in the undamaged state must be altered. Let G_{ij} denote this metric. G and $\underline{G}$ are related by

$$\underline{G}_{ij} = F^{i}_{k} F^{j}_{l} G^{kl}$$

Therefore, if

$$G^{kl} = \sum_{1 \leq a \leq 3} f^{k}_{a} f^{l}_{a}$$

then, as required,

$$\underline{G}_{ij} = \delta_{ij}$$

Of course, formally the stress-strain relation so obtained are identical to (9-4); only the coordinate system is different. Although indices can be raised and lowered using δ_{ij} and δ^{ij} so that the distinction between the tensor types disappears, equation (5-2) must now be rewritten to reflect the fact that the metric in the undamaged state is not Euclidean. For example, E^2 written in terms of covariant components is $E_{ij} E_{jl} G^{kl}$.

2.10 For some applications, the finite deformation theory developed so far may be unnecessarily complex; this section develops simple linear equations for permanent memory materials in a viscoelastic range.

The most systematic approach to deriving linear theories starts with the exact equations and introduces suitable approximations, such as infinitesimal strains, which lead to linearity. Although this approach will not be followed here, it is worthwhile to understand what would be involved in carrying it out. The first linearization of equation (5-2) would not be to introduce infinitessimal strains directly. Instead, one would start with the theory of section 9 and assume infinitesimal strains about the damaged traction-free configuration. This would be an example of infinitesimal strain superposed on finite strain (see [5], §3.1) since the strains needed to deform the original stress-free state to the damaged state might be finite. Although this theory would be linear in the strain measures, using it would still require solution of the nonlinear boundary value problem which determines the damaged traction-free configuration.

A second degree of approximation is to assume that the strains in the damaged traction free configuration are infinitessimal. This is the theory which will be developed in this section. This theory is

(a) linear in D and in the infinitesimal strains E_{ij}

and more importantly,

(b) the total strain in the damaged body is the sum of the strain needed to bring the original rest state to the damaged traction-free configuration and the strain needed to bring the damaged traction free configuration to the present configuration.

This type of theory is appropriate when damage is caused by strains of the order of 10% to 20%, and subsequent testing involves strains of

a few percent.(*) Thus, since the permanent sets are rarely large, the total strains are certainly less than 25% and infinitesimal strain theory is adequate.

Damage is defined as before in terms of the maximum tensile strain:

$$d = - \max [\varepsilon(t) + |\varepsilon(t)|]/2$$

where $|\varepsilon(t)|$ is the matrix absolute value function. The stress-strain relations are

$$\sigma = \sigma(d, \varepsilon)$$

The ideas of isotropic damaging developed in section 2, and the use of Rivlin's expansions as in section 4 are still appropriate. However, in using the expansion, all terms which are nonlinear in ε are dropped. The question then arises of which of the other terms ought to be retained. Because the characterization applies to a viscoelastic range, d must be several times larger than ε; thus, it is reasonable to retain some products of d and ε and even perhaps some powers of d even though d is much smaller than one. Also, it is desirable to model anisotropy in the damaged material. On the other hand, the experimental data do not really justify a very complex theory. In view of these considerations, the following equation is proposed for an initially isotropic material with Lamé constants λ and μ:

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* The subsequent strains must be smaller to insure that the material remains in a viscoelastic range.

$$\sigma = f_1 I + f_2 d + f_3 \varepsilon + f_4(\varepsilon d + d\varepsilon) \tag{10-1}$$

where

$$\begin{aligned} f_1 &= a_{11}\,\mathrm{tr}d + \lambda(1 + a_{12}\,\mathrm{tr}d)\,\mathrm{tr}\varepsilon + a_{13}\,\mathrm{tr}(\varepsilon d) \\ f_2 &= a_{21} + a_{22}\,\mathrm{tr}\varepsilon \\ f_3 &= 2\mu(1 + a_{31}\,\mathrm{tr}d) \\ f_4 &= a_{41} \end{aligned} \tag{10-2}$$

Thus, all terms separately linear in d and ε are included. Strictly speaking, this theory is not consistent: if terms of order $d\varepsilon$ are included, terms of order d^2 should be as well. We justify this inconsistency by the poor quality of the experimental data, and we justify retaining terms involving $d\varepsilon$ by the need to model possible anisotropy in the damaged material.

A discussion of (10-1) and (10-2) along the lines of section 6 may be useful. First, when d = 0, (10-1) reduces to the initial characterization $\sigma = \lambda\,\mathrm{tr}\varepsilon I + 2\mu\varepsilon$. The constants a_{11} and a_{21} model permanent set, since when $\varepsilon = 0$, (10-1) becomes

$$\sigma = a_{11}(\mathrm{tr}d)\, I + a_{21} d$$

Thus, a_{11} corresponds to a hydrostatic stress, and a_{21} to a stress proportional to d. These are the stresses needed to restore the original rest configuration.

The remaining constants in (10-2) model modulus changes a_{12} and a_{31} model isotropic damage in which an initially isotropic material remains isotropic after damage, but with possibly reduced moduli. Note that a_{12} models a change in bulk response only.

a_{22} corresponds to terms (c), section 6. When $tr\ \varepsilon = 0$, that is, for a volume preserving deformation, this term does not contribute to the damaged stress-strain relation. Thus, it models an anisotropic change in bulk response only; the shear moduli are unchanged.

The other constants, a_{13} and a_{41}, result in orthorhombic behavior in the damaged state. These terms modify both bulk and shear properties.

Alternatively, (10-1) and (10-2) can be derived from a damage-dependent strain energy function $W = W(d, \varepsilon)$ as in section 7. To derive the type of theory developed here, W should be at most quadratic in ε and linear in d. Also, W should reduce to $\frac{1}{2}\lambda\ (tr\varepsilon)^2 + \mu tr(\varepsilon^2)$ when $d = 0$. Thus,

$$W = \frac{1}{2}\lambda\ (tr\varepsilon)^2\ (1 + a_{12}\ trd) + \mu tr\varepsilon^2\ (1 + a_{31}\ trd)$$

$$+ a_{21}\ tr(\varepsilon d) + a_{41}\ tr\varepsilon^2 d = a_{11}\ (tr\varepsilon)\ (trd) + a_{22}\ tr(\varepsilon d)\ tr\varepsilon$$

Substituting into $\sigma_{ij} = \dfrac{\partial w}{\partial \varepsilon_{ij}}$ leads to

$$\sigma = [a_{11}\ trd + \lambda(1 + a_{12}\ trd)\ tr\varepsilon + a_{22}\ tr(\varepsilon d)]\ I$$

$$+ (a_{21} + a_{22}\ tr\varepsilon)\ d + 2\mu(1 + a_{31}\ trd)\ \varepsilon + a_{41}\ (\varepsilon d + d\varepsilon) \qquad (10\text{-}3)$$

This is the same as equation (10-1) and (10-2), except that $a_{13} = a_{22}$.

To rewrite (10-3) in the form $\sigma_{ij} = \sigma_{ij}^{(0)} + C_{ijkl}\,\varepsilon_{kl}$ choose axes so that d is in diagonal form. Then the elastic constants of the damaged material are:

$$C_{1111} = \lambda(1 + a_{12}\,\mathrm{trd}) + 2\mu(1 + a_{31}\,\mathrm{trd}) + (a_{13} + a_{22} + 2a_{41})\,d_{11}$$

$$C_{2222} = \lambda(1 + a_{12}\,\mathrm{trd}) + 2\mu(1 + a_{31}\,\mathrm{trd}) + (a_{13} + a_{22} + 2a_{41})\,d_{22}$$

$$C_{3333} = \lambda(1 + a_{12}\,\mathrm{trd}) + 2\mu(1 + a_{31}\,\mathrm{trd}) + (a_{13} + a_{22} + 2a_{41})\,d_{33}$$

$$C_{1122} = \lambda(1 + a_{12}\,\mathrm{trd}) + a_{22}(d_{11} + d_{22})$$

$$C_{1133} = \lambda(1 + a_{12}\,\mathrm{trd}) + a_{22}(d_{11} + d_{33})$$

$$C_{2233} = \lambda(1 + a_{12}\,\mathrm{trd}) + a_{22}(d_{22} + d_{33})$$

$$C_{1212} = 2\mu(1 + a_{31}\,\mathrm{trd}) + 2a_{41}(d_{11} + d_{22})$$

$$C_{1313} = 2\mu(1 + a_{31}\,\mathrm{trd}) + 2a_{41}\,(d_{11} + d_{33})$$

$$C_{2323} = 2\mu(1 + a_{31}\,\mathrm{trd}) + 2a_{41}\,(d_{22} + d_{33})$$

These nine constants are characteristic of an orthorhombic material. The crystal axes coincide with the principal axes of d. However, (10-3) does not represent a general orthorhombic material. Indeed, as in section 5 , (10-3) cannot have cubic symmetry since $c_{1111} = c_{2222} = c_{3333}$ forces $d_{11} = d_{22} = d_{33}$, which leads to isotropy.

2.11 REFERENCES

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List of Symbols for Chapter III

$\underset{\sim}{C}$	Green's tensor
d_p	Duration of a process
$d_e(\cdot,\cdot)$	Viscosity metric
$d_m(\cdot,\cdot)$	Maximum metric
$d_p(\cdot,\cdot)$	Weighted Lebesgue metric
$d_r(\cdot,\cdot)$	Relaxation metric
$d_G(\cdot,\cdot)$	Configuration metric
$d_\Sigma(\cdot,\cdot)$	Metric on the state space
$d(\cdot,\cdot)$	History metric
$d_r(\cdot,\cdot)$	Relaxation History metric
$d_D(\cdot,\cdot)$	Damage history metric
$d_G(\cdot,\cdot)$	Configuration history metric
$\hat{D}$	Damage function
$\underset{\sim}{D}$	Damage tensor
D^*	Relative damage history
D_E	Damage resulting from a constant strain E
$\mathscr{D}$	Damage space
$\mathfrak{D}$	Damage functional
g	Function; influence function
G	Configuration
$\hat{G}$	Configuration mapping
$\underset{\sim}{G}$	Tensor valued kernel

G^*	Stress-free configuration
$G_{(t)}$	Freeze for time t at configuration G
$\mathcal{G}$	Configuration space
I	Strain invariant
I_d	Volume dilation
I_γ	Octahedral shear strain
$\lvert\cdot\rvert$	Absolute value
$\lVert\cdot\rVert_p$	p^{th} order Lebesgue average
$\hat{\lambda}$	State relaxation mapping
λ_o	Virgin state
$\bar{n}$	Unit normal
P_G^σ	Set of processes starting at state σ and ending in configuration G
Π	Set of deformation processes
Π_G	Set of processes that start at configuration G
R	Rotation tensor; strain rate
$\hat{\rho}$	Evolution function
$\bar{\rho}$	History evolution function
s	Stress value of the sample; dummy variable
$\hat{S}$	Stress mapping
$\bar{S}(H)$	Stress corresponding to a history H
$\mathfrak{S}$	Response functional
$\hat{\sigma}$	Relaxed state
Σ	State space
Σ_G	G section of Σ

$\Sigma_{G,D}$	G,D section of Σ
$\Sigma_{G\ rel}$	Set of relaxed states in the G configuration
Σ_{rel}	Set of relaxed states
$\Sigma_{rel.D.}$	Set of relaxed states with damage D
t	Time

CHAPTER III. GENERAL THEORY

3.1 Introduction

The hysteresis effect associated with Permanent Memory materials can be visualized as a weakening or damaging of the material as demonstrated in Sulijoadikusumo and Fitzgerald [1]. Quinlan and Fitzgerald [2 , 3] have presented this concept in a constitutive equation and incorporated it into a characterization applicable to highly filled polymeric materials. They did not, however, define an order relation on damage. This was accomplished by Fitzgerald [4].

This present paper expands upon and refines the work of Fitzgerald [4]. The structure developed by Noll [5] and Fitzgerald [4] will be used herein.

The topology on the state space Σ is first developed, followed by a discussion on some metrics possible for this space. It is shown how Fitzgerald's [4] definition of comparability of damage defines a natural order on $\Sigma_{\underset{\sim}{1}\ \mathrm{rel}}$, the set of relaxed states in the unity configuration.

Results of experiments by Penny [6] motivate the discussion of residual stress and permenent set. Finally, it is shown how Noll's theory of semi-elastic material elements can be readily extended to permanent memory materials with semi-elastic range, PMSE.

3.2 Bodies, Processes and States

Let $\mathcal{G}$ be any set. A function P of the type

$$P: [0,d_p] \to \mathcal{G}, \quad d_p \varepsilon R^+$$

will be called a *process* with values in $\mathcal{G}$. The number d_p will be called the *duration* of the process and the values

$$P^i = P(0), \quad P^f = P(d_p)$$

will be referred to as the *initial and final values* of the process.

If $G\varepsilon\mathcal{G}$ and $t\varepsilon R^+$, we define a process $G_{(t)}$: $[0,t] \to \mathcal{G}$ is defined by

$$G_{(t)}(r) = G \quad \text{for} \quad r\varepsilon[0,t]$$

and call it the *freeze* of duration t at G.

Let P_1 and P_2 be two processes such that $P_i^f = P_2^i$. A new process $P_i * P_2$ of duration $d_{p_1} + d_{p_2}$ is then defined by

$$(P_1 * P_2)(t) = \begin{cases} P_1(t) & \text{if } t\varepsilon[0,d_{p_1}] \\ P_2(t-d_{p_1}) & \text{if } t\varepsilon[d_{p_1}, d_{p_1} + d_{p_2}]. \end{cases} \qquad (3\text{-}1)$$

$P_1 * P_2$ is called the *continuation* of P_1 with P_2. Continuation is an associative process, i.e.,

$$(P_1 * P_2) * P_3 = P_1 * (P_2 * P_3)$$

if

$$P_1^f = P_2^i = (P_2 * P_3)^i \quad \text{and} \quad P_2^f = (P_1 * P_2)^f = P_3^i.$$

Processes of duration zero are not excluded. Of course, all such processes are of the form $G_{(0)}$, i.e., they are freezes of duration zero. It is clear that for any process P,

$$P = P^i_{(0)} * P = P * P^f_{(0)}$$

Definition 1: A *body element* is a triple $(\mathcal{T}, \mathcal{G}, \Pi)$ where $\mathcal{T}$ is a finite-dimensional real vector space, $\mathcal{G}$ is a closed and connected subset of $\mathrm{Sym}^+(\mathcal{T}, \mathcal{T}^*)$ and Π is a class of processes with values in $\mathcal{G}$ which satisfies the following conditions:

(P1) Any freeze at any $G \in \mathcal{G}$ belongs to Π.

(P2) If P belongs to Π, so does every segment of P.

(P3) Π is closed under continuation, i.e., if $P_1, P_2 \in \Pi$ and $P_1^f = P_2^i$, then $P_1 * P_2 \in \Pi$.

(P4) Any two elements $G_1, G_2 \in \mathcal{G}$ can be connected by a process in Π, i.e., there is at least one $P \in \Pi$ such that $G_1 = P^i$ and $G_2 = P^f$.

When speaking about a body element $\mathcal{T}$, it is understood that $\mathcal{T}$ is endowed with a structure defined by the prescription of $\mathcal{G}$ and Π. The elements of $\mathcal{G}$ will be called the *configuration* of $\mathcal{T}$ and the processes in Π the *deformation processes* for $\mathcal{T}$.

If it is assumed that $\mathcal{G}$ is arcwise connected, one such class Π that satisfies (P1) - (P4) is the class of all processes P that are continuous and have a piecewise continuous derivative $\dot{P}$ in the sense that $\dot{P}(t)$ exist for all but a finite number of $t\in[0,d_p]$ and $\dot{P}$ has left and right limits for all $t\in[0,d_p]$.

Most of the development herein utilizes the concept of *intrinsic stress*. This describes the contact forces acting on an element intrinsically, that is, without any reference to a frame space. Objectivity is thus automatically satisfied throughout. Full details may be had in Noll [5]. However, what has been said here will suffice for the reader to understand what follows.

The primitive notion of state space will be used to fully describe the condition, configuration, and response to deformation processes of a material body element. Permanent memory effects, i.e., damage, will also be fully described by the state of the material element.

It is now necessary to generalize Noll's definition of material element to a permanent memory (PM) material element.

Definition 2: A PM *material element* is an octuple $(\mathcal{T},\mathcal{G},\Pi,\Sigma,\hat{G},\hat{S},\hat{\rho},\hat{D})$ which satisfies Axioms I to VI below. The objects of the octuple are:

a) $(\mathcal{T},\mathcal{G},\Pi)$ is the underlying body element, Definition 1, of the material element.

b) Σ is a set, called the *state space* of the material element.

c) $\hat{G}$ is a mapping $\hat{G}$: $\Sigma \to \mathcal{G}$ from the state space Σ into the *configuration space* $\mathcal{G}$.

d) $\hat{S}$ is a mapping $\hat{S}$: $\Sigma \to \mathcal{S}$ from the state space Σ into the *stress space* $\mathcal{S} = \text{Sym}(\mathcal{T}^*, \mathcal{T})$.

e) $\hat{\rho}$ is a mapping $\hat{\rho}$: $(\Sigma \times \Pi)_{fit} \to \Sigma$, whose domain

$$(\Sigma \times \Pi)_{fit} = \{(\sigma,P) | \sigma \in \Sigma,\ P \in \Pi,\ P^i = \hat{G}(\sigma)\} \tag{3-2}$$

is the set of all process pairs such that the state "fits" the initial configuration of the process.

f) $\hat{D}$ is a mapping $\hat{D}$: $\Sigma \to \mathcal{D}$ from the state space Σ onto the *damage space* $\mathcal{D} \subset \text{Sym}+(\mathcal{T}, \mathcal{T}^*)$.

The elements σ of Σ are the possible states of $\mathcal{T}$. The value $\hat{G}(\sigma)$ is the configuration and $\hat{S}(\sigma)$ is the (intrinsic) stress determined by the state σ. The function $\hat{\rho}$ is the *evolution function* of $\mathcal{T}$ whose value $\hat{\rho}(\sigma,P)$ is the state reached by the element if, starting from the state σ, it is subjected to the deformation process P. The value $\hat{D}(\sigma)$ is the damage associated with the state σ.

In addition, let us now define after Noll [5]:

· $\Pi_G = \{P \in \Pi \,|\, P^i = G\}$ as the set of all deformation processes beginning at the configuration $G \in \mathcal{G}$.

· $\Sigma_G = \{\sigma \in \Sigma \,|\, \hat{G}(\sigma) = G\}$ as the set of all states that fit the configuration $G \in \mathcal{G}$. Σ_G is called the *G section* of Σ.

· The *response functional* $\mathfrak{S}$ of the element is defined by

$$\mathfrak{S} = \hat{S}o\hat{\rho}: \quad (\Sigma \times \Pi)_{fit} \to \mathcal{S}.$$

If $\sigma \in \Sigma$ and $P \in \Pi_{\hat{G}(\sigma)}$, then $\mathfrak{S}(\sigma,P)$ is the stress produced by the process P when the initial state was σ.

Both Π_G and Σ_G are disjoint for different values of G, moreover,

$$\Pi = \bigcup_{G\in\mathcal{G}} \Pi_G$$

$$\Sigma = \bigcup_{G\in\mathcal{G}} \Pi_G.$$

Also, $(\Sigma \times \Pi)_{fit}$ is the disjoint union of $\Sigma_G \times \Pi_G$, $G\in \mathcal{G}$.

The foregoing definitions establish the state σ of the PM material element $\mathcal{T}$ as a primitive concept, embodying within it the configuration, intrinsic stress, and damage of that state. It is a fundamental property of mappings, that a point in the domain is carried to only one point in the range. That is, if $\hat{A}$ is a mapping, and $\sigma = \tau \Rightarrow \hat{A}(\sigma) = \hat{A}(\tau)$ or equivalently: with σ, τ elements of the domain of $\hat{A}$.
$\hat{A}(\sigma) \neq \hat{A}(\tau) \Rightarrow \sigma \neq \tau$.

Thus, applying this observation to the mappings presented in Definition 2, we find that we cannot have differing stresses, configurations, or damages associated with the same stress state.

The first two axioms of Noll [5] flow directly from the intuitive meaning of the evolution function $\hat{\rho}$. The first states that the state reached after a process must fit the final configuration. The second expresses the fact that when the element is subjected to a process P_1 and then to a process P_2, it must reach the same state as when it is subjected to the continuation $P_1 * P_2$ of P_1 by P_2.

Axiom I: For all $(\sigma,P)\in(\Sigma \times \Pi)_{fit}$.

$$\hat{\rho}(\sigma,P)\in \Sigma_p f, \quad \text{i.e.,} \quad \hat{G}(\hat{\rho}(\sigma,P)) = P^f \qquad (3\text{-}3)$$

It follows from Axiom I that $G: \Sigma \to \mathcal{G}$ must be surjective because by condition (P4) of Definition 1, for any $\sigma\epsilon\Sigma$ and any $G\epsilon\mathcal{G}$, there must be process $P\epsilon\Pi$ such that $(\sigma,P)\epsilon(\Sigma \times \Pi)_{fit}$ and $P^f = G$.

Axiom II: If P_1, P_2 , $\sigma\epsilon\Sigma p_1^i$ and $P_1^f = P_2^i$, then

$$\hat{\rho}(\sigma,(P_1 * P_2) = \hat{\rho}(\hat{\rho}(\sigma,P_1),P_2) \ . \tag{3-4}$$

The immediate consequence of this from the definition of the response functional above, is

$$\mathfrak{S}(\sigma,P_1 * P_2) = \mathfrak{S}(\hat{\rho}(\sigma,P_1),P_2) \ . \tag{3-5}$$

The next axiom is as basic to PM materials as it is to Noll's semi-elastic materials. It expresses the assumption that there must be some operational way to distinguish between states. That is, if two states are different but fit the same configuration, there must be some process starting at these states, which produces different stress responses.

Axiom III: For any configuration $G\epsilon\mathcal{G}$, if the states $\sigma_1, \sigma_2 \ \epsilon\Sigma_G$ and the stress response

$$\mathfrak{S}(\sigma_1,P) = \mathfrak{S}(\sigma_2,P) \tag{3-6}$$

for every deformation process $P\epsilon\Pi_G$, then the states $\sigma_1 = \sigma_2$.

The topology for PM material elements is now introduced following Noll [ ]. The natural uniformity of the stress space is utilized to develop this topology.

Definition 3: Let $\mathcal{T}$ be a PM material element with response functional $\mathcal{F}$. For every $G \in \mathcal{G}$, we call the natural uniformity of of the G section, the coarsest uniformity on Σ_G which renders the mappings

$$\mathcal{F}(\cdot, P): \Sigma_G \to \mathcal{G}$$

uniformly continuous for all $P \in \Pi_G$. We call the natural topology of the state space $\Sigma = \cup\{\Sigma_G \mid G \in \mathcal{G}\}$ the sum of the topologies on the sections Σ_G induced by the natural uniformities of these sections.

The natural topology on Σ is such that all sections $\Sigma_G, G \in \mathcal{G}$ are both open and closed in Σ. Hence $\hat{G}: \Sigma \to \mathcal{G}$ is trivially continuous, because it is constant on each section:

The following result from Noll [ ] characterises the natural topology of Σ.

Proposition 1: A net in Σ converges to $\sigma \in \Sigma$ if and only if ${}^{(i)}G_i$ belongs eventually to some fixed section Σ_G, ${}^{(ii)}\sigma \in \Sigma_G$ and ${}^{(iii)}$ $\lim \mathcal{F}(\sigma_i, P) = \mathcal{F}(\sigma, P)$ $\forall$ $P \in \Pi_G$

The following postulate is made to avoid mathematical complications.

Axiom IV: The sections $\Sigma_G, G \in \mathcal{G}$, with the natural uniformity discribed above, are complete spaces.

If a material body element is in some state σ and is then frozen in the corresponding configuration $G(\sigma)$, eventually a "relaxed" state will be reached.

The next Axiom of Noll's expresses the existence of this state.

Axiom V: For all $\sigma \in \Sigma$, the limit

$$\lim_{t\to\infty} \hat{\rho}(\sigma, \hat{G}(\sigma)_{(t)}) = \hat{\lambda}(\sigma) \tag{3-7}$$

exists (and is unique by Proposition 2).

The mapping $\hat{\lambda} : \Sigma \to \Sigma$ determined by Axiom V will be called the *state relaxation mapping* with the members of its range

$$\Sigma_{rel} = \hat{\lambda}(\Sigma)$$

being called *relaxed states*.

Clearly, $\hat{G}(\hat{\lambda}(\sigma)) = \hat{G}(\sigma)$, and since $\hat{G}$ is surjective, the restriction $\hat{G}\big|_{\Sigma_{rel}}$ of $\hat{G}$ to Σ_{rel}, is also surjective.

If Proposition 1 is applied to the net $t \to \hat{\rho}(\sigma, \hat{G}(\sigma)_{(t)})$ and equation (5) observed, it is seen that (7) is equivalent to

$$\lim_{t\to\infty} \mathfrak{G}(\sigma, G_{(t)} * P) = \mathfrak{G}(\hat{\lambda}(\sigma), P)$$

for all $P \in \Pi_G$, where $G = \hat{G}(\sigma)$.

To simplify notation in the following sections the following definitions are required:

Definition 4: The set of all relaxed states with damage, D, will be denoted by $\Sigma_{rel\ D}$

Therefore

$$\Sigma_{G\ rel\ D} = \{\sigma\epsilon\Sigma \mid \sigma\epsilon\Sigma_{rel} \text{ and } \hat{D}(\sigma) = D\}.$$

Definition 5: The set of all relaxed states in the configuration G will be denoted by $\Sigma_{G\ rel}$.

Therefore,

$$\Sigma_{G\ rel} = \{\sigma\epsilon\Sigma \mid \hat{G}(\sigma) = G \text{ and } \sigma\epsilon\Sigma_{rel}\}.$$

Thus, the quantity $\Sigma_{\underset{\sim}{1}\ rel}$ in the following section is the set of relaxed states in the unity configuration

3.3 Partial Order on the Damage Space

The concept of permanent memory or damage is embodied in the idea that permanent material property changes affect the state of a material. These changes are assumed to occur as a result of the past history of deformation of the material element. Primary causes of damage are the magnitude of the maximum deformation gradient [ ] the duration of the deformation process at the maximum, the temperature at which the maximum occurs [8], and, if applicable, a slow (relative to the deformation processes) healing which can decrease or, in the limit, eliminate the damage effects [2, 3].

However, we are not here limited to effects which weaken the material. Permanent material property change can also be effected by processes which increase the material modulus. Processes embraced by this study thus include aging, cold rolling and fiber orientation.

In short, the concept of damage embodies all permanent material changes that are brought about by isothermal strain histories.

The physical idea of damage will be incorporated into the mathematical structure by means of a partial order on the damage space, $\mathfrak{D}$.

The treatment is essentially that of Fitzgerald [4]. The partial order on $\mathcal{D}$ will be defined in terms of the natural partial order on the stress space $\mathcal{S}$.

The stress space $\mathcal{S}$ can be regarded as the set of self-adjoint bounded linear operations on the three dimensional real vector space. We can thus apply the treatment of Brown and Page [9] to develop its partial order.

Definition 6: An operator $T \varepsilon \mathcal{S}$ is said to be positive if and only if $(Tn, n) > 0$ for all vectors n.

We shall denote by $\mathcal{S}^+$ the set of all positive operators in $\mathcal{S}$.

Definition 7: Given $T, S \varepsilon \mathcal{S}$ we shall write $T \leq S$ if and only if $S - T \varepsilon \mathcal{S}^+$.

Obviously $T < S$ if and only if $(Tn, n) < (Sn, n)$ for all vectors n [9].

The stress space $\mathcal{S}$, is thus partially ordered, and we shall use this to define a partial order on $\Sigma_{G\ rel}$.

Definition 8: Given $\sigma, \tau \varepsilon \Sigma_{G\ rel}$ we say $\sigma < \tau$ if and only if $\mathcal{T}(\sigma,P) > \mathcal{T}(\tau,P)$ for all $P \varepsilon \pi_G$.

Proposition 2: The set $\Sigma_{G\ rel}$ is partially ordered.

Proof: Reflexivity: Choose $\sigma \varepsilon \Sigma_{G\ rel}$ for a particular process we have $\mathcal{T}(\sigma, P)$ as the stress response.

By the reflexive property on

$$\mathcal{T}(\sigma,P) < \mathcal{T}(\sigma, P)$$

The process $P \varepsilon \pi G$ is arbitrary, therefore $\sigma < \sigma$ by definition.

Antisymmetry: Choose $\sigma, \tau \varepsilon \Sigma_{G\ rel}$ such that $\sigma < \tau$ and $\tau < \sigma$

Therefore $\mathfrak{S}(\sigma, P) \leq \mathfrak{S}(\tau, P)$ and $\mathfrak{S}(\tau, P) \leq \mathfrak{S}(\sigma, P)$

both equations holding for all $P \in \Pi_G$.

Thus for a particular $P \in \Pi_G$

$$\mathfrak{S}(\sigma, P) = \mathfrak{S}(\tau, P)$$

by the antisymmetry of $\mathfrak{S}$.

Since $P \in \Pi_G$ is arbitrary, we thus have $\sigma = \tau$ by Axiom III.

Transitivity: Choose $\sigma, \tau, \gamma \in \Sigma_{G\ rel}$ with $\sigma \leq \tau$ and $\tau \leq \gamma$.

Therefore $\mathfrak{S}(\sigma, P) \leq \mathfrak{S}(\tau, P)$ and $\mathfrak{S}(\tau, P) \leq \mathfrak{S}(\gamma, P)$

both equations holding for all $P \in \Pi_G$. Thus for a particular process P we have by the transitivity of $\mathfrak{S}$,

$$\mathfrak{S}(\sigma, P) \leq \mathfrak{S}(\gamma, P)$$

Since $P \in \Pi_G$ is arbitrary we thus have by definition 6 ,

$$\sigma \leq \gamma$$

Thus $\Sigma_{G\ rel}$ is partially ordered.

We now enlist the aid of the following axiom, which is a generalisation of Nolls definition of Semi - Elastic elements.

Axiom VI: The mapping $\hat{\mathfrak{D}}|_{\Sigma_{G\ rel}} : \Sigma_{G\ rel} \rightarrow \mathfrak{D}$ is bijective.

Becasue of the partial order on $\Sigma_{G\ rel}$, this axiom induces a natural partial order on $\mathfrak{D}$.

Definition 9: Given $\sigma, \tau \in \Sigma_{G\ rel}$

$\hat{\mathfrak{D}}(\sigma) \leq \hat{\mathfrak{D}}(\tau)$ if and only if $\sigma \leq \tau$.

Thus $\mathfrak{D}$ is partially ordered by each bijective mapping $\hat{\mathfrak{D}}|_{\Sigma_{G\ rel}}$. The only step left is to make each partial order on $\mathfrak{D}$ compatible. This is accomplished by placing a restriction on the choice of the mapping $\hat{D}$.

Restriction: If $D_1, D_2 \in \mathfrak{D}$ are such that there exists $\sigma, \tau \in \Sigma_{G\,rel}$ and $\sigma', \tau' \in \Sigma_{G'\,rel}$ with $D_i = \hat{D}(\sigma) = \hat{D}(\tau)$ and $D_2 = \hat{D}(\sigma') = \hat{D}(\tau')$

Then if $\hat{D}(\sigma) \leq \hat{D}(\tau)$

we must have $\hat{D}(\sigma') \leq \hat{D}(\tau')$

Proposition 3: The damage space $\mathfrak{D}$ is partially ordered.

Proof: Reflexivity: Choose $D \in \mathfrak{D}$. Then, for an arbitrary choice of $G \in \mathfrak{G}$, there exists $\sigma \in \Sigma_{G\,rel}$ such that $\hat{D}(\sigma) = D$.

Now, $\sigma \leq \sigma$.

Therefore $\hat{D}(\sigma) \leq \hat{D}(\sigma)$ and $D \leq D$.

Antisymmetry: Choose $D_1, D_2 \in \mathfrak{D}$, such that $D_1 \leq D_2$ and $D_2 \leq D_1$. For an arbitrary choice of $G \in \mathfrak{G}$, there exist states $\sigma_1, \sigma_2 \in \Sigma_{G\,rel}$ such that $\hat{D}(\sigma_1) = D_1$ and $\hat{D}(\sigma_2) = D_2$. Therefore by hypothesis,

$\hat{D}(\sigma_1) \leq \hat{D}(\sigma_2)$ and $\hat{D}(\sigma_2) \leq \hat{D}(\sigma_1)$

Thus $\sigma_1 \leq \sigma_2$ and $\sigma_2 \leq \sigma_1$, which implies that $\sigma_1 = \sigma_2$

therefore $D_1 = D_2$

Transitivity: Choose D_1, D_2 $D_3 \in \mathfrak{D}$, such that $D_1 \leq D_2$ and $D_2 \leq D_3$.

For an arbitrary choice of $G \in \mathfrak{G}$, there exist states $\sigma_1, \sigma_2, \sigma_3 \in \Sigma_{G\,rel}$, such that $\hat{D}(\sigma_1) = D_1$, $\hat{D}(\sigma_2) = D_2$ and $\hat{D}(\sigma_3) = D_3$.

We have that

$\hat{D}(\sigma_1) \leq \hat{D}(\sigma_2)$ and $\hat{D}(\sigma_2) \leq \hat{D}(\sigma_3)$

Therefore

$\sigma_1 \leq \sigma_2$ and $\sigma_2 \leq \sigma_3$

giving that $\sigma_1 \leq \sigma_3$ Thus $D_1 \leq D_3$.

We can see that a natural consequence of the above is that if $D_1 \leq D_2$, then there exists a configuration G and states $\sigma_1, \sigma_2 \in \Sigma_{G\,rel}$, such that, $\hat{D}(\sigma_1) = D_1$ and $\hat{D}(\sigma_2) = D_2$. Then, since $\sigma_1 \leq \sigma_2$, we have that

$\mathcal{G}(\sigma_2, \Sigma) \leq \mathcal{G}(\sigma_1, \Sigma) \quad \forall \Sigma \in \Pi_G$

A subset $\mathfrak{D}^*$ of a partially ordered set $\mathfrak{D}$ is said to be a chain if for any two elements D_1, $D_2 \in \mathfrak{D}^*$ one has either $D_1 < D_2$ or $D_2 < D_1$. Thus the restriction of the partial ordering relation to $\mathcal{D}^*$ yields a total ordering.

Elements of a chain in the damage space $\mathfrak{D}$ are said to be comparable. Two damage states not elements of the same chain, are said to be incomparible.

Definition 10: For any $\sigma, \tau \in \Sigma_{G\,rel}$, the degree of damage $\hat{D}(\tau) < \hat{D}(\sigma)$ if and only if $\hat{D}(\tau) \leq \hat{D}(\sigma)$

and

$$\langle \mathfrak{S}(\sigma, P) n, n \rangle < \langle \mathfrak{S}(\tau, P) n, n \rangle$$

for at least one $P \in \Pi_G$ and one direction n.

Lemma: If $\sigma, \tau \in \Sigma_{G\,rel}$, then $\hat{D}(\sigma) = \hat{D}(\tau)$ if and only if

$$\mathfrak{S}(\sigma, P) = \mathfrak{S}(\tau, P) \quad \forall \; P \in \Pi_G$$

Proof: If $\hat{D}(\sigma) = \hat{D}(\tau)$ then by the reflexivity of the partial order on $\mathfrak{D}$ we have $\hat{D}(\sigma) \leq \hat{D}(\tau)$ and $\hat{D}(\tau) \leq \hat{D}(\sigma)$

hence $\mathfrak{S}(\tau, P) \leq \mathfrak{S}(\sigma, P) \quad \forall \; P \in \Pi_G$

and $\mathfrak{S}(\sigma, P) \leq \mathfrak{S}(\tau, P) \quad \forall \; P \in \Pi_G$

For a particular process P, both equations hold, therefore by the antisymmetric property of the partial order on $\mathfrak{S}$,

$$\mathfrak{S}(\sigma, P) = \mathfrak{S}(\tau, P) \quad \forall \; P \in \Pi_G$$

The converse follows by symmetry.

We may note here, in order to emphasize the compatability of this treatment with that of Fitzgerald[4] , that

$$\mathfrak{S}(\sigma, P) = \mathfrak{S}(\tau, P) \quad \forall \; P \in \Pi_G$$

is equivalent to his

$$\langle \mathfrak{S}(\sigma, P) n, n \rangle = \langle \mathfrak{S}(\tau, P) n, n \rangle \; \forall \; P \in \Pi_G$$

and for all vectors n.

The proof is obvious from a restatement of the latter equation:

$$\langle \left(\mathfrak{P}(\sigma, \mathcal{P}) - \mathfrak{P}(\tau, \mathcal{P}) \right) n, n \rangle = 0.$$

Lemma: The set $\Sigma_{G \text{ rel } D}$ has at most one element if and only if the mapping $\hat{G}|_{\Sigma_{\text{rel } D}}$ is injective.

Proof: Let $\sigma, \tau \in \Sigma_{\text{rel } D}$ with $\hat{G}(\sigma) = \hat{G}(\tau) = G$

hence $\sigma, \tau \in \Sigma_{G \text{ rel } D}$

and $\sigma = \tau$ by hypothesis.

Let $\sigma, \tau \in \Sigma_{G \text{ rel } D}$.

Since, $\hat{G}(\sigma) = \hat{G}(\tau)$ it follows, by the injectivity of $\hat{G}|_{\Sigma_{\text{rel } D}}$ that $\sigma = \tau$.

Lemma: If $\hat{A}$ and $\hat{B}$ are relations (not necessarily functions) with the same domain X, then $\hat{A}|_{X_D}$ is injective if and only if $\hat{B}|_{X_A}$ is injective, where

$$X_A = \{ x \in X \mid \hat{A}(x) = A \}$$

$$X_B = \{ x \in X \mid \hat{B}(x) = B \}$$

Proof: Let $\sigma, \tau \in X_A$ with $\hat{B}(\sigma) = \hat{B}(\tau) = B$.

hence $\sigma, \tau \in X_B$ and $\hat{A}(\sigma) = \hat{A}(\tau) = A$

hence $\sigma = \tau$ by hypothesis.

hence $\hat{B}|_{X_A}$ is injective. This is a reciprocity lemma.

The converse follows by symmetry.

Proposition 4: The set $\Sigma_{G \text{ rel } D}$ of relaxed states with a given configuration G and damage D, has at most one element, which is unique.

Proof: The proof of the first part follows from both lemmas and Axiom VI. Uniqueness follows from the observation that, if either

$\hat{G}(\sigma) \neq \hat{G}(\tau)$ or $\hat{D}(\sigma) \neq \hat{D}(\tau)$, then $\sigma \neq \tau$.

We now wish to axiomatize the fact that every relaxed state in the unity configuration must be comparible to some initial material state. The treatment follows that of Penny [6].

Axiom VII: The elements $\sigma \in \Sigma_{\underset{\sim}{1}\,rel}$, form a space of chains which are not everywhere disjoint. In particular, every state in $\Sigma_{\underset{\sim}{1}\,rel}$ is comparible to a preferred relaxed state λ_o, called the virgin state of the PMM element.

We normalise the damage associated with the virgin state λ_o to $\underset{\sim}{1}$. That is, $\hat{D}(\lambda_o) = \underset{\sim}{1}$.

Axiom VIII: For each chain $X \subset \Sigma_{\underset{\sim}{1}\,rel}$ there exist states σ_x, τ_x such that $\sigma \leq \sigma_x$ and $\tau_x \leq \sigma \quad \forall\, \sigma \in X$

Proposition 5: There exist states σ^* and τ^* such that

$$\sigma \leq \sigma^* \quad \text{and} \quad \tau^* \leq \sigma \quad \forall\, \sigma \in \Sigma_{\underset{\sim}{1}\,rel}$$

Proof: The proof follows from the application of Zorn's lemma.

The maximal element σ^* and the minimal element τ^* are comparible to each other and also to every element in $\Sigma_{\underset{\sim}{1}\,rel}$. The space $\Sigma_{\underset{\sim}{1}\,rel}$ can thus be pictured as being a space of chains originating at λ_o and terminating at either σ^* or τ^*.

If $\sigma \in \Sigma_{rel}$, we distinguish between $\underset{\sim}{1} \leq \hat{D}(\sigma)$ and $\hat{D}(\sigma) \leq \underset{\sim}{1}$, by saying in the former case that the state is damaged, and in the latter case, that the state exhibits strain history hardening.

The following proposition shows that a damaged relaxed state in the unity configuration cannot have tensile residual stresses.

Proposition 6: If $\sigma \in \Sigma_{\underset{\sim}{1}\,rel}$ then

a) $\underset{\sim}{1} \leq \hat{D}(\sigma) \Rightarrow \hat{S}(\sigma) \leq 0$

b) $\hat{D}(\sigma) \leq 1 \Rightarrow 0 \leq \hat{S}(\sigma)$

Proof: To prove a) we compare σ with $\lambda_o \in \Sigma_{\underset{\sim}{1}\ rel}$ Since $\hat{D}(\lambda_o) = 1$, we have $\hat{D}(\lambda_o) < \hat{D}(\sigma)$. Thus, by definition, $\lambda_o < \sigma_1$, and so also $S(\sigma,P) < S(\lambda_o, P)$ for all $P \in \pi_G$. Upon restricting P to being a freeze at the identity configuration, we find that

$$\hat{S}(\sigma) \leq \hat{S}(\lambda_o)$$

But since λ_o is the virgin state and thus has no residual stress, we find that $\hat{S}(\sigma) < 0$. The case of initial stress, S_o, follows readily. The proof of b) follows by symmetry.

3.4 Subclasses of Permanent Memory Materials

Within the general designation of Permanent Memory Material, three main subclasses can be discerned. These are Thixotropic solids, Thixotropic fluids and Permanent Memory Materials with Semi-Elastic range (PMSE).

Thixotropic Solid: A PMM element is said to be a Thixotropic solid element if, given $\sigma \in \Sigma_{rel,\ S=0}$, $\hat{G}(\sigma) = G_o$ for some G_o.

We can see that given a stress this material adopts a unique relaxed configuration independent of the damage.

Proposition 7: For a Thixotropic solid element, if $\sigma \in \Sigma_{rel,\ S=0}$, then $\hat{G}(\sigma) = 1$

Proof: Since $\lambda_o \in \Sigma_{rel\ S=0}$, and since $\hat{G}(\lambda_o) = \underset{\sim}{1}$ the conclusion follows.

A thixotropic solid, therefore, cannot have permanent set nor residual stress. The damage sustained by this material occurs entirely in the "viscous" phase and not at all in the "structural" phase.

Thixotropic Fluid: A PM material element is a Thixotropic fluid element if and only if for all $\sigma \in \Sigma_{rel}$ whose configuration $G = \hat{G}(\sigma)$ satisfied $\det(\tilde{G})^2 = 1$

where $\tilde{G}$ is the associated bilinear form for G, $\hat{S}(\sigma)=0$

3.5 Permanent Memory Semi-Elastic Materials (PMSE)

A PM element is a PMSE element if and only if $\hat{G}\big|_{\Sigma_{rel\,S}}$ is injective.

Proposition 8: For a PMSE element, if $\sigma, \tau \in \Sigma_{rel\,S}$ then

$$\hat{G}(\sigma) = \hat{G}(\tau) \Longrightarrow \hat{D}(\sigma) = \hat{D}(\tau)$$

Proof: Choose $\sigma, \tau \in \Sigma_{rel\,S}$

By hypothesis $\hat{G}\big|_{\Sigma_{rel\,S}}$ is injective, thus

$\hat{G}(\sigma) = \hat{G}(\tau) \Leftrightarrow \sigma = \tau \Rightarrow \hat{D}(\sigma) = \hat{D}(\tau)$.

Corollary: For a PMSE element, if $\sigma, \tau \in \Sigma_{rel\,S}$ then

$\hat{D}(\sigma) \neq \hat{D}(\tau) \Rightarrow \hat{G}(\sigma) \neq \hat{G}(\tau)$.

This implies that relaxed states with different damages have different stress free configurations.

Corollary: For a PMSE element, if $\sigma \in \Sigma_{rel\,S=0}$ with $\hat{D}(\sigma) \neq \underset{\sim}{1}$, then σ must have permanent set.

Proof: This follows by letting $\tau = \lambda_o$ in the above.

Therefore

$\hat{D}(\sigma) \neq \underset{\sim}{1} \Rightarrow \hat{G}(\sigma) \neq \underset{\sim}{1}$.

Proposition 9: For a PMSE element, if $\sigma, \tau \in \Sigma_{G\,rel}$ then

$\hat{D}(\sigma) = \hat{D}(\tau) \Leftrightarrow \sigma = \tau \Leftrightarrow \hat{S}(\sigma) = \hat{S}(\tau)$.

Proof: All has been proven except, $\hat{S}(\sigma) = \hat{S}(\tau) \Rightarrow \sigma = \tau$, and this follows from the injectivity of $\hat{S}\big|_{\Sigma_{G\,rel}}$ (see reciprocity Lemma)

Lemma: For a PMSE element, if $\sigma \in \Sigma_{\underset{\sim}{1}\,rel\,D}$ with $D \neq 1$ then the stress free configuration is not the identity configuration.

Proof: Let $\sigma \in \Sigma_{\underset{\sim}{1}\,rel\,D}$ with $D \neq 1$. Suppose $\underset{\sim}{1}$ is the stress free configuration of σ. Comparing σ with λ_o, the virgin state, we see that, $\sigma, \lambda_o \in \Sigma_{rel,\,S=0}$.

Since $\hat{G}(\sigma) = \hat{G}(\lambda_o) = \underset{\sim}{1}$, we have by proposition 8 that

$$\hat{D}(\sigma) = \hat{D}(\lambda_o) = 1$$

This contradicts the hypothesis of the proposition. Therefore $\underset{\sim}{1}$ is not the stress force configuration of σ.

Proposition 10: For a PMSE element, if $\sigma \in \Sigma_{\underset{\sim}{1}\ \text{rel}\ D}$ with $1 < D$ then

$$< \hat{S}(\sigma)n,\ n > \leq 0$$

with strict inequality holding for at least one direction n.

Proof: By Proposition 5

$$\hat{S}(\sigma) \leq 0.$$

However, since $\underset{\sim}{1}$ is not the stress free configuration, inequality cannot hold everywhere in the above equation.

We can see from the above that for a strain history hardening material, there must be some tensile residual stress.

We may also note from proposition 9 that if a fluid is PMSE, its damage must be manifested as a change in volume.

PMSE solid: A PMSE solid is a PMSE material such that $\hat{D}|_{\Sigma_{\text{rel}\ s}}$ is injective.

Proposition 11: The following statements are equivalent and thus all hold for a PMSE solid.

i) The partial order on $\mathcal{D}$ defines a partial order on $\Sigma_{\text{rel}\ s}$.

ii) The mapping $\hat{D}|\Sigma_{\text{rel}\ s}$ is injective

iii) Relaxed states with equal damage and stress have equal configurations.

iv) The set $\Sigma_{\text{rel}\ s\ D}$ contains at most one element.

Proof: [i) $\Rightarrow$ ii)] Let the partial order in question be:

If $\sigma, \tau \in \Sigma_{\text{rel s}}$ then

$\sigma \leq \tau \Leftrightarrow \hat{D}(\sigma) \leq \hat{D}(\tau)$.

Consider $\sigma, \tau \in \Sigma_{\text{rels}}$ with $\hat{D}(\sigma) = \hat{D}(\tau)$.

By the reflexive property of the partial order on $\mathfrak{D}$ we have

$$\hat{D}(\sigma) \leq \hat{D}(\tau) \quad \text{and} \quad \hat{D}(\tau) \leq \hat{D}(\sigma).$$

Therefore $\sigma \leq \tau$ and $\tau \leq \sigma$, and thus $\sigma = \tau$ by hypothesis.

$\therefore$ $\hat{D}|_{\Sigma_{\text{rel s}}}$ is injective.

[ii) $\Rightarrow$ iii)] Let $\sigma, \tau \in \Sigma_{\text{rel s D}}$

Since $\hat{D}(\sigma) = \hat{D}(\tau)$, it follows by hypothesis that $\sigma = \tau$.

Therefore $\hat{G}(\sigma) = \hat{G}(\tau)$.

[iii) $\Rightarrow$ iv)] Let $\sigma, \tau \in \Sigma_{\text{rel s D}}$. By hypothesis $\hat{G}(\sigma) = \hat{G}(\tau)$, and thus by Axiom $\sigma = \tau$.

[iv) $\Rightarrow$ i)] We must show that $\Sigma_{\text{rel s}}$ is partially ordered by $\leq$ where this is defined for $\sigma, \tau \in \Sigma_{\text{rel s}}$, as

$\sigma \leq \tau \Leftrightarrow \hat{D}(\sigma) \leq \hat{D}(\tau)$.

Reflexivity: Let $\sigma \in \Sigma_{\text{rel s}}$ Therefore $\hat{D}(\sigma) \in \mathfrak{D}$ and thus

$\hat{D}(\sigma) \leq \hat{D}(\sigma)$.

$\therefore$ $\sigma \leq \sigma$

Antisymmetry: Let $\sigma, \tau \in \Sigma_{\text{rel s}}$ with $\sigma \leq \tau$ and $\tau \leq \sigma$

$\therefore$ $\hat{D}(\sigma) \leq \hat{D}(\tau)$ and $\hat{D}(\tau) \leq \hat{D}(\sigma)$

$\therefore$ $\hat{D}(\sigma) = \hat{D}(\tau)$

But $\sigma, \tau \in \Sigma_{\text{rel s D}}$, and thus by hypothesis $\sigma = \tau$.

Transitivity Let $\sigma, \tau, \gamma \in \Sigma_{\text{rel s}}$ with $\sigma \leq \tau$ and $\tau \leq \gamma$.

Therefore $\hat{D}(\sigma) \leq \hat{D}(\tau)$ and $\hat{D}(\tau) \leq \hat{D}(\gamma)$

$\therefore$ $\hat{D}(\sigma) \leq \hat{D}(\gamma)$

$\therefore$ $\sigma \leq \gamma$

Therefore $\Sigma_{rel\,s}$ is partially ordered and consequently statements i) to iv) are equivalent and by definition must hold for a PMSE solid.

We now induce a natural partial order on the configurations of relaxed states with equal stresses.

Definition 11: If $\sigma, \tau \in \Sigma_{rel\,s}$ then

$$\hat{G}(\sigma) \leq \hat{G}(\tau) \Leftrightarrow \sigma \leq \tau.$$

Proposition 12: For a PMSE solid element, the set $\hat{G}(\Sigma_{rel\,s})$ is partially ordered.

Proof: This follows from the injectivity of $\hat{G}|_{\Sigma_{rel\,s}}$

Corollary: If $\sigma, \tau \in \Sigma_{rel\,s}$ then

$$\hat{G}(\sigma) \leq \hat{G}(\tau) \Leftrightarrow \hat{D}(\sigma) \leq \hat{D}(\tau).$$

The proof is immediate, thus for a PMSE solid, the stress free states are ordered by the damage.

We see therefore that incompressible damaging PMSE solid fluids cannot exist.

We see from Proposition 11 that for a PMSE solid $\Sigma_{rel\,s}$ s = o,D has at most one element. The class of materials represented by having many elements, is the class of fluids that can sustain damage in the viscous component of their structure. PMSE fluids can exist. but any damage sustained must be manifested as a change in volume. The material classes of Mullins elastic and inviscid Fluids are those classes such that every state is relaxed. The material classes of SE solids and SE fluids encompass together the Noll SE class of materials. These classes are characterised by their inability to sustain damage.

3.6 Metrics on the State Space

We now turn to a consideration of suitable metrics on the state space Σ. A metric, $d(\cdot,\cdot)$, on Σ can be chosen if and only if, the

stress functional

$$\mathcal{S}(\cdot,\Omega) : (\Sigma, \mathcal{d}(\cdot,\cdot)) \longrightarrow \mathcal{S}$$

is continuous.

Let us now consider a restriction on this metric.

Let $\quad \mathcal{d}_\Sigma(\cdot,\cdot) = \mathcal{d}_r(\cdot,\cdot) + \mathcal{d}_G(\cdot,\cdot) + \mathcal{d}_D(\cdot,\cdot),$

where

$\mathcal{d}_r(\cdot,\cdot)$ is the *relaxation* metric,

$\mathcal{d}_G(\cdot,\cdot)$ is the *configuration* metric,

$\mathcal{d}_D(\cdot,\cdot)$ is the *damage* metric

and where

a) $\mathcal{d}_r(\hat{\sigma}_1,\hat{\sigma}_2) = 0$ for all $\hat{\sigma}_1, \hat{\sigma}_2 \in \Sigma_{rel}$

b) $\mathcal{d}_D(\sigma_1,\sigma_2) = 0$ whenever $\hat{D}(\sigma_1) = \hat{D}(\sigma_2)$

c) $\mathcal{d}_G(\sigma_1,\sigma_2) = 0$ whenever $\hat{G}(\sigma_1) = \hat{G}(\sigma_2)$.

Proposition 13: Every relaxed state $\sigma \in \Sigma_{rel}$ is an accumulation point of

$$(\Sigma, \mathcal{d}_\Sigma(\cdot,\cdot)) .$$

Proof: Choose $\hat{\sigma} \in \Sigma_{rel}$.

There exists a $\sigma \in \Sigma$ such that

$$\hat{\sigma} = \hat{\lambda}(\sigma) = \lim_{t\to\infty} \hat{\rho}(\sigma,\hat{G}(\sigma)_{(t)}).$$

Construct the net $i \to \sigma_i$ such that

$$\sigma_i = \hat{\rho}(\sigma,\hat{G}(\sigma)_{(i)})$$

$$\therefore \lim_{i\to\infty} \sigma_i = \hat{\sigma}$$

and

$$d_{\Sigma}(\sigma_i,\hat{\sigma}) \to 0 \quad \text{as } i \to \infty .$$

$\therefore$ σ_i is eventually in every arbitrary neighborhood of $\hat{\sigma}$.

$\therefore$ $\hat{\sigma}$ is an accumulation point of Σ.

Proposition 14: The relaxation metric $d_r(\cdot,\cdot)$ satisfies

$$d_r(\sigma,\hat{\sigma}) = d_r(\sigma,\hat{\tau})$$

where $\hat{\sigma} = \hat{\lambda}(\sigma)$, for all $\sigma\epsilon\Sigma$ and for all $\hat{\tau}\epsilon\ \Sigma_{rel}$.

Proof: Let $\sigma\epsilon\ \Sigma$ and $\hat{\lambda}(\sigma) = \hat{\sigma}$, then

$$d_r(\sigma,\hat{\sigma}) \leq d_r(\sigma,\hat{\tau}) + d_r(\hat{\tau},\hat{\sigma})$$

$$= d_r(\sigma,\hat{\tau}) \qquad \text{by a).}$$

Also

$$d_r(\sigma,\hat{\tau}) \leq d_r(\sigma,\hat{\sigma}) + d_r(\hat{\sigma},\hat{\tau})$$

$$= d_r(\sigma,\hat{\sigma}) \qquad \text{by a).}$$

$\therefore$ $d_r(\sigma,\hat{\sigma}) = d_r(\sigma,\hat{\tau})$ for all $\hat{\tau}\epsilon\Sigma_{rel}$

Q.E.D.

From the above observations it is seen that $d_r(\cdot,\cdot)$ only measures differences in relaxation between states.

Some particular forms for these metrics will now be discussed. First, a preliminary definition from Noll [5].

Definition 12: A function H of the type $H:R^+ \to \mathcal{G}$ will be called a *history* with values in $\mathcal{G}$. Its value

$$H(0) = H^f$$

will be called the *final value* of H. Let H_1 give rise to the state σ_1 and H_2 give rise to the state σ_2 in what follows.

Let us now define

$$\bar{d}'(H_1,H_2) = \bar{d}_\Sigma(\sigma_1,\sigma_2),$$

$$\bar{d}'_r(H_1,H_2) = \bar{d}_r(\sigma_1,\sigma_2),$$

$$\bar{d}'_G(H_1,H_2) = \bar{d}_G(\sigma_1,\sigma_2) \quad \text{and}$$

$$\bar{d}'_D(H_1,H_2) = \bar{d}_D(\sigma_1,\sigma_2)$$

These metrics could assume the following particular forms [50]:

$$\bar{d}'_D(H_1,H_2) = \bar{d}_m(H_1,H_2) + \bar{d}_e(\dot{H}_1(0),\dot{H}_2(0)) + \bar{d}_p(H_1,H_2), \quad p > 2$$

where the norm

$$||H(s)||_e = [\mathrm{tr}H(s)^2]^{1/2},$$

$$\bar{d}_e(\dot{H}_1(0),\dot{H}_2(0)) = ||\dot{H}_1(0) - \dot{H}_2(0)||_e,$$

$$\bar{d}_m(H_1,H_2) = \left| \operatorname*{ess\,sup}_{s\in[0,\infty)} ||H_1(s)||_e - \operatorname*{ess\,sup}_{s\in[0,\infty)} ||H_2(s)||_e \right|,$$

and

$$\bar{d}_p(H_1,H_2) = \left[\int_0^\infty h^p(s)||H_1(s) - H_2(s)||_e^p \, ds \right]^{1/p}.$$

The quantity h(s) is the fading memory influence function of Coleman and Noll.

This form would imply that damage was dependent on the difference in the maximums (d_m), the difference in the instantaneous viscosity (d_e), and a weighted average of the histories (d_p).

The following could also be chosen

$$d'_G(H_1,H_2) = d_e(H_1(0),H_2(0))$$

and

$$d'_r(H_1,H_2) = d_2(H_1,H_2)$$

$$= [\int_0^\infty h^2(s)||H_1(s) - H_2(s)||_e^2 \, ds]^{\frac{1}{2}}.$$

The above represent possible forms for the metrics d'_G, d'_r and d'_D. Again it must be noted that the particular form chosen must be such as to cause the stress functional to be continuous.

3.7 Accessibility

It is necessary to borrow again from Noll [5] to develop the idea of accessibility.

The set of all states that can be reached from a state

is

$$\{\hat{\rho}(\sigma,P)|P\varepsilon\Pi_{\hat{G}(\sigma)}\} .$$

Let us now define

$$\Sigma_\sigma = \text{cl}\{\hat{\rho}(\sigma,P)|P\varepsilon\Pi_{\hat{G}(\sigma)}\}$$

where the closure is in the sense of the natural topology defined above. If a state $\tau \in \Sigma_\sigma$, then the state τ is accessible from the state σ. Thus τ is accessible from σ, if every neighborhood of τ contains states that can actually be reached from σ.

Transitivity of accessibility is equivalent to the statement that, if $\tau \in \Sigma_\sigma$ then $\Sigma_\tau \subset \Sigma_\sigma$. This is Noll's Proposition 13.1.

The next proposition, Noll [5], is a direct consequence of (7) and the transitivity of accessibility.

Proposition 15: For every $\sigma \in \Sigma$, if $\tau \in \Sigma_\sigma$, then $\lambda(\tau) \in \Sigma_\sigma$, i.e., if τ is accessible from σ, so is $\lambda(\tau)$. In particular, $\lambda(\sigma) \in \Sigma_\sigma$. We can apply Noll's Proposition 10.5 to PM materials.

Proposition 16: In a PM material element two states of equal damage are mutually accessible.

This proposition tells us about accessibility within the sets Σ_D, however, the question of accessibility between states of differing damages is as yet untouched. Fitzgerald [4] again supplies the appropriate axiom.

Axiom IX: Given any two comparable relaxed states $\hat{\sigma}$ and $\hat{\tau}$ then with $\hat{D}(\hat{\sigma}) > \hat{D}(\hat{\tau})$ we say that $\hat{\sigma}$ is accessible from τ, $\hat{\sigma} \in \Sigma_{\hat{\tau}}$, and $\hat{\tau}$ is not accessible from $\hat{\sigma}$, $\hat{\tau} \notin \Sigma_{\hat{\sigma}}$. Incomparabe relaxed states are mutually inaccessible, i.e.,

$$\hat{\sigma} \notin \Sigma_{\hat{\tau}} \text{ and } \hat{\tau} \notin \Sigma_{\hat{\sigma}}$$

when $\hat{\sigma}$ and $\hat{\tau}$ are incomparable.

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Aging effects are neither included nor excluded in this treatment since (9)orders damage only in relaxed configurations. See Fitzgerald [4] for aging considerations on PMSE materials.

3.8 CONSTITUTIVE EQUATIONS

The previously referenced papers [2, 3] provided a first approximation to a constitutive equation compatible with the concept of damage presented herein.

Hufferd [7] has shown a computational scheme for obtaining explicit characterizations from experimental data. His scheme is now being tested under an Air Force Rocket Propulsion Laboratory contract with United Technology Corporation.

Lee [8] has adapted the general constitutive equation to include transient temperature effects and has successfully modeled the results of some simultaneous thermal change - straining data obtained from Lockheed Propulsion Company on their Air Force STV program.

Thiokol Chemical Corporation in collaboration with W. L. Hufferd is currently using an extension of [2, 3] to model chemical aging effects.

For the relaxed states discussed herein, the fading memory terms of [2, 3] vanish and the stress tensor, S, as a function of damage, D, and a strain measure, C, become

$$S = U(D,C)$$

Equicontinuity of S in D and C with the stress free virgin state then permit the above equation to assume the form

$$S = U_1(D, C)C + U_2(D, C)D$$

For a positive damage, D, the material element in a stress-free configuration, C*, then reduces the above equation to

$$U_2(D,C^*)D = -U_1(D,C^*)C^*$$

From [2] it can be shown that U_2 has a unique inverse, thus

$$D = -U_2^{-1}(D, D^*)\, U_1(D,C^*)C^*$$

For strain histories in the semi-elastic range of a material then,

$$S = U_1(D, C)C - U_2^{-1}(D,C^*)\, U_1(D,C^*)\, U_2(D,C)C^*$$

For a pseudo-linear semi-elastic range approximation

$$U_1(D,C) = U_1(D)$$

$$U_2(D\text{-}C) = U_1(D)$$

hence, $S = U_1(D)\, C - U_2^{-1}(D)\, U_1(D)\, U_2(D)C^*$

For isotropic materials, the fourth order tensor moduli U_1, U_2 have the usual scalar decomposition so that we then have

$$S = U_1(D)\,(C\text{-}C^*) = U_1(D)\,\bar{C}$$

where $\bar{C} = C\text{-}C^*$ is effectively the strain measure taken with respect to the permanent set configuration.

It is this strain measure that is used in the figures of Sulijoadikusumo [1] where $U_1(D)$ shows the "softening" effect of damage which accords with our definition of damage or herein.

3.9 CONCLUSIONS

A definition and ordering of the damage tensor has been given mathematical form. The definition is operational in that laboratory tests follow naturally from the definition.

Several sub-classes of materials, within the Permanent Memory class are elucidated, Thixotropic Solids, Thixotropic Fluids, and PMSE materials.

The PMSE class exhibits permanent set in the stress free configuration after damage. PMSE fluids exhibit a volume increase after damage.

The PMSE class exhibits residual stresses in the idem configuration after damage. This residual stress is nowhere tensile.

For strain histories which do not cause the damage to increase, PMSE

materials behave as Noll's SE group.

The PM thixotropic classes exhibit damage through their "viscous components". That is, damage can only be observed during a process departing from a relaxed state, transient tests.

PMSE materials exhibit damage through changes in their "structural" or "elastic" phase, the damage may be observed in the relaxed state.

PMSE materials may also exhibit damage effects in their "viscous" phases.

The classes herein described, in addition to the above, include Mullins elasticity and reduce to Noll's SE and (non-linear and linear) elasticity as well as linear fluids on the limit.

It is hoped that a proper mathematical structure reflecting observed mechanical phenomena has been herein developed.

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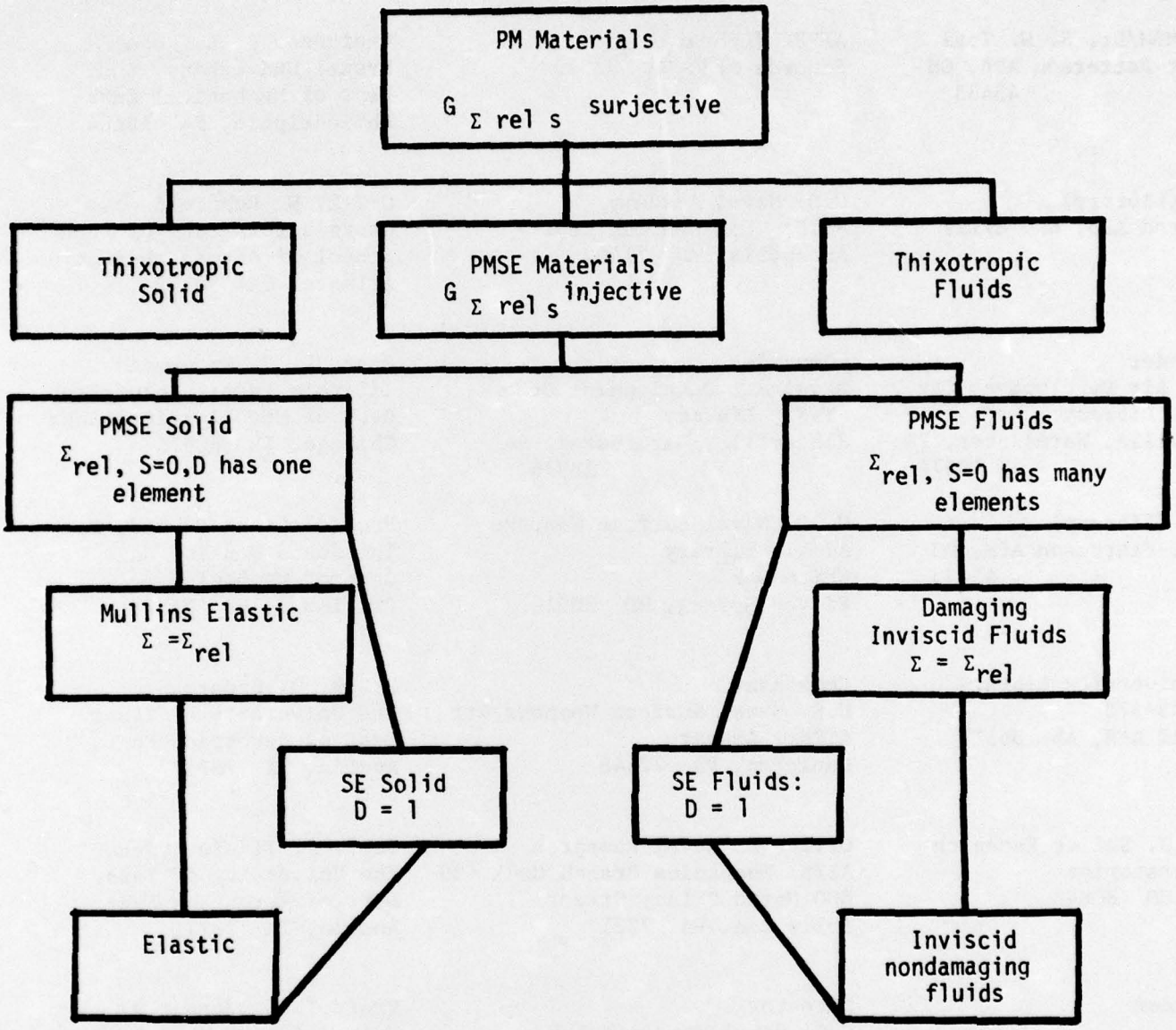

FIGURE 1

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